Name. GROUP 1 Sample 2270 Midterm 1, S2010

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Applied Linear Algebra 2270-2

Sample Midterm Exam 1
Thursday, 25 Feb 2010

Instructions: This in-class exam is designed for 50 minutes. No tables, notes, books or calculators allowed.

1. (Inverse of a matrix) Supply details for two of these:

a. If A and B are $n \times n$ invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.

b. If possibly non-square matrices A and B satisfy $AB = I$, then $B\mathbf{x} = \mathbf{0}$ cannot have infinitely many solutions.

c. Give an example of a 4×3 system having a unique solution.

$\boxed{A.}$

$A^{-1}A = I$; $B^{-1}B = I$ — by Theorem 2.4.6, Matrix multiplied by its inverse is Identity.

$(AB)^{-1} = B^{-1}A^{-1}$ — given

$AB(AB)^{-1} = B^{-1}BA^{-1}A$ — multiply both sides by AB. Remember by Thm. 2.3.6 multiplication is associative.

$AB(AB)^{-1} = I$ — By step 1; $I \cdot I = I$

$A^{-1}AB(AB)^{-1} = A^{-1}I$ — multiply both sides by A^{-1}

$IB(AB)^{-1} = A^{-1}I$ — by step 1

$B(AB)^{-1} = A^{-1}$ — Identity multiplied by a matrix = the matrix

$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$ — multiply by B^{-1}

$I(AB)^{-1} = B^{-1}A^{-1}$ — by step 1

$(AB)^{-1} = B^{-1}A^{-1}$ — by step 7

$\boxed{C}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 3 nullity = 0

In $m \times n$ matrix, if rank $(A) = n$ there is a unique solution.

Theorem 1.3.2

$\boxed{B}$

$AB = I$ — given

$B\vec{x} = \vec{0}$ — given

$AB\vec{x} = A\vec{0}$ — multiply both sides by A

$I\vec{x} = \vec{0}$ — left by step 1; right by matrix · zero vector = zero vector.

$\vec{x} = \vec{0}$ — Identity multiplied by $\vec{x} = \vec{x}$ (see part A).

We discover $B\vec{x} = \vec{0}$ cannot have infinitely many solutions because it has a unique solution $\vec{x} = \vec{0}$.

Please staple this page to your solution. Write your initials on all pages.

Sample Exam Problem 2

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Let $\boldsymbol{A}$ be a 3×3 matrix. Let:

$$\boldsymbol{F} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Assume F is obtained from $\boldsymbol{A}$ by the following sequential row operations:
(1) Swap rows 1 and 3;
(2) Add -2 times row 2 to row 3;
(3) Add 3 times row 1 to row 2;
(4) Multiply row 2 by 5.

a. Write a matrix multiplication formula for F in terms of explicit elementary matrices and the matrix $\boldsymbol{A}$. (80%)
b. Find $\boldsymbol{A}$. (20%)

$$\boldsymbol{E}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \boldsymbol{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{E}_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \boldsymbol{E}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \boldsymbol{E}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \boldsymbol{E}_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a.

$$\boldsymbol{A} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\boldsymbol{E}_3^{-1}\boldsymbol{E}_4^{-1}F$$

b.

$$\boldsymbol{A} = \begin{pmatrix} -6 & -6 & -58/5 \\ -3 & -3 & -29/5 \\ 1 & 1 & 2 \end{pmatrix}$$

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PART 1

A $M \times N$ HOMOGENEOUS SYSTEM $A\vec{x}=\vec{0}$ HAS ∞-MANY SOLUTIONS WHEN $M<N$

$M<N$ IMPLIES AT LEAST 1 FREE VARIABLE

AT LEAST 1 FREE VARIABLE IMPLIES NULLITY >0

NULLITY >0 IMPLIES ∞-MANY SOLUTIONS

#3

PART 2

$$\left|\begin{matrix} 1 & b-c & a & -a \\ 1 & c & -a & a \\ 2 & b & a & a \end{matrix}\right|$$

$$\left|\begin{matrix} 1 & b-c & a & -a \\ 0 & b-2c & 2a & -2a \\ 2 & b & a & a \end{matrix}\right|$$

$$\left|\begin{matrix} 1 & b-c & a & -a \\ 0 & b-2c & 2a & -2a \\ 0 & b-2c & a & -3a \end{matrix}\right|$$

$$\left|\begin{matrix} 1 & b-c & a & -a \\ 0 & b-2c & 2a & -2a \\ 0 & 0 & a & a \end{matrix}\right|$$

UNIQUE SOLUTION

$$\boxed{\begin{matrix} a \neq 0 \\ b-2c \neq 0 \end{matrix}}$$

NO SOLUTION

$$\begin{matrix} a \neq 0, \\ b-2c = 0 \end{matrix}$$

∞-MANY SOLUTIONS

$$a = 0$$

$$\begin{matrix} 1 & b-c & a & -a \\ 0 & b-2c & 0 & -4a \\ 0 & 0 & a & a \end{matrix}$$

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1. $\boxed{A}$ Problem 4

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} 2a & a \\ 2c & c \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} 3a & a+b \\ 3c & c+d \end{bmatrix}$$

$$\begin{bmatrix} c-3a & d-(a+b) \\ a-3c & b-(c+d) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} -3a \quad + c \quad &= 0 \\ a \quad -3c \quad &= 0 \\ -a \quad -b \quad + d &= 0 \\ b - c - d &= 0 \end{aligned}$$

$$\begin{bmatrix} -3 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{aligned} a &= 0 \\ b &= t_1 \\ c &= 0 \\ d &= t_1 \end{aligned} \Rightarrow A = \begin{bmatrix} 0 & t_1 \\ 0 & t_1 \end{bmatrix} = t_1 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$\boxed{C}$ $(A+B)(A-B) = A^2 - B^2$

$$\begin{aligned} A^2 + BA - AB - B^2 &= A^2 - B^2 \\ BA - AB &= 0 \\ BA &= AB \end{aligned}$$

$\therefore$ A and B commute

$\boxed{D}$ Let A be a 3×2 matrix and B a 2×3 matrix. Explain using matrix algebra
and the three possibilities why the 3×3 matrix $C = AB$ cannot satisfy $\mathrm{rref}(C) = I$.

Ans: Since B is 2×3, $\dim(\mathrm{Im}(B)) \le 2$, so by rank-nullity,
$\dim(\mathrm{Ker}(B)) \ge 1$, which means there is a zero vector $\vec{x}$ satisfying
$B\vec{x} = \vec{0}$ for that vector.

$$C\vec{x} = AB\vec{x} = A\cdot\vec{0} = \vec{0}$$
$$C\vec{x} = \vec{0}$$

implies C is not the Identity. ✓

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = A \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + A$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} 2a & a \\ 2c & c \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} 3a & a+b \\ 3c & c+d \end{bmatrix}$$

$$\left|\begin{array}{l} c = 3a \\ d = a+b \\ a = 3c \\ b = c+d \end{array}\right| \quad \begin{array}{l} \rightarrow a = 0 \\ d = b \\ \rightarrow c = 0 \\ b = d \end{array}$$

$$A = \begin{bmatrix} 3c & c+d \\ 3a & a+b \end{bmatrix} = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix} = k\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

(b) $\text{rref}(c) = I \implies C\vec{x} = \vec{b}$

$$\rightarrow C\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\rightarrow AB\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A\vec{x}$ not possible. $(3\times 2)(3\times 1)$

$$\rightarrow A\vec{y} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let $B\vec{x} = \vec{y}$ $(2\times \cancel{3})(\cancel{3\times 1})$

$$\rightarrow A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A\vec{y} = \vec{b}$ has no solutions.

$\therefore \text{rref}(c) \neq I$ if $C = AB$

(c) $(A+B)(A-B) = A^2 - B^2$

$$AA - AB + BA - BB = A^2 - B^2$$
$$A^2 - AB + BA - B^2 = A^2 - B^2$$
$$(A^2 - B^2) - AB + BA = A^2 - B^2$$
$$-AB + BA = 0$$
$$BA = AB \checkmark$$

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5. **(Geometry and linear transformations)**

Part I. Classify $T(\mathbf{x}) = A\mathbf{x}$ geometrically as scaling, projection onto line L, reflection in line L, pure rotation by angle θ, rotation composed with scaling, horizontal shear, vertical shear. [60%]

a. $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\theta = 90°$ $\Rightarrow$ Scale by 2 c.c. rotation by $90°$

b. $A = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \Rightarrow$ horizontal shear; $k=5$

c. $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}$; $u_1 = 0$, $u_2 = 1$ $\Rightarrow$ Projection onto Line L.

d. $A = \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}$; $u_1 = \frac{\sqrt{3}}{2}$, $u_2 = 1/2$ $\Rightarrow$ Projection onto line L.

e. $A = \frac{1}{2}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$; $a = -\sqrt{3}/2$, $b = 1/2$, $a^2+b^2 = \frac{3}{4}+\frac{1}{4} = 1$ $\Rightarrow$ Reflection in line L.

Part II. Give details. [40%]

f. Define reflection in a line L in $\mathcal{R}^3$.

g. Display the matrix A of a projection onto a line L in $\mathcal{R}^3$.

h. Define rotation clockwise by angle θ in $\mathcal{R}^2$.

f. $T(\vec{x}) = A\vec{x} \Rightarrow \mathrm{Ref}(\vec{x}) = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x} \Rightarrow$ in $\mathbb{R}^3$ $A = \begin{bmatrix} 2u_1^2-1 & u_2u_1 & u_3u_1 \\ u_1u_2 & 2u_2^2-1 & u_3u_2 \\ u_1u_3 & u_2u_3 & 2u_3^2-1 \end{bmatrix}$

g. $\mathrm{Proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u} \Rightarrow A = \begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & u_2^2 & u_2u_3 \\ u_1u_3 & u_2u_3 & u_3^2 \end{bmatrix}$ in $\mathbb{R}^3$

h. $T(\vec{x}) = A\vec{x}$

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{ in } \mathbb{R}^2$$

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