Name. KEY 2270 Midterm 1, S2010

Applied Linear Algebra 2270-2

Midterm Exam 1

Thursday, 25 Feb 2010

Instructions: This in-class exam is designed for 50 minutes. No tables, notes, books or calculators allowed. You will be given extra time to complete the exam. If a result is true, provide a proof, else explain why it is false.

1. **(Inverse of a matrix)** Supply details.

   (a) [40%] If A and B are 10×10 invertible matrices, then
   $$(AB^2A)^{-1} = (AB)^{-1}(BA)^{-1}.$$

   (b) [30%] Give an example of a 2×2 system $A\mathbf{x} = \mathbf{b}$ containing symbol k which has a unique solution except for $k = 1, 2, 3, \ldots, 10$ [first ten positive integers].

   (c) [30%] If possibly non-square matrices A and B satisfy $AB = I$, then $A\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$.
   Warning: Because inverses may not exist, a proof cannot use symbols like A^{-1}.

(a) $(AB^2A)^{-1} = ((AB)(BA))^{-1} = (BA)^{-1}(AB)^{-1}$ by $(CD)^{-1} = D^{-1}C^{-1}$ Theorem.
Result is false.

(b) $\begin{pmatrix} 1 & 0 \\ 0 & f(k) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ where $f(k) = (k-1)(k-2)\cdots(k-10)$

(c) The equation $B\vec{y} = \vec{0}$ has a unique solution, because $B\vec{y} = \vec{0}$ $\Rightarrow$ $AB\vec{y} = A\vec{0}$ $\Rightarrow$ $I\vec{y} = \vec{0}$ $\Rightarrow$ $\vec{y} = \vec{0}$.
Assume $A\vec{x} = \vec{0}$. Solve $\vec{x} = B\vec{y}$ for $\vec{y}$, possible because B has rank $= n$, nullity $= 0$, meaning $\mathrm{rref}(B) = I$ in the square case or n leading ones in the non-square case. Then

$$\begin{aligned} A\vec{x} &= AB\vec{y} \\ &= I\vec{y} \\ &= \vec{y} \end{aligned}$$

implies $\vec{0} = \vec{y}$

implies $B\vec{0} = B\vec{y}$

$\vec{0} = \vec{x}$

completing the proof.

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2. (Elementary Matrices) Let A be a 3×3 matrix. Let

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assume B is obtained from A by the following sequential row operations:

(1) Multiply row 2 by -2
(2) Swap rows 2 and 3
(3) Add -4 times row 2 to row 3
(4) Add 5 times row 1 to row 2.

(a) [50%] Write a formula for B in terms of explicit elementary matrices multiplied against the matrix A.

(b) [50%] Find A.

(a) $$B = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

(b) $$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ inverse operations in reverse order

$$\begin{pmatrix} 1 & 1 & 0 \\ -5 & -5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{combo}(1,2,-5)$$

$$\begin{pmatrix} 1 & 1 & 0 \\ -5 & -5 & 1 \\ -20 & -20 & 4 \end{pmatrix} \quad \text{combo}(2,3,4)$$

$$\begin{pmatrix} 1 & 1 & 0 \\ -20 & -20 & 4 \\ -5 & -5 & 1 \end{pmatrix} \quad \text{swap}(2,3)$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 10 & 10 & -2 \\ -5 & -5 & 1 \end{pmatrix} \quad \text{mult}(2,-\tfrac{1}{2})$$

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3. (RREF method)

(a) [20%] Explain using linear algebra terminology why a 10×15 linear homogeneous system $A\mathbf{x} = \mathbf{0}$ has two nonzero solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, with $\mathbf{x}_1$ not a scalar multiple of $\mathbf{x}_2$.

(b) [80%] Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & b+c & a \\ 3 & 2b+4c & a \\ -1 & -b-c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a^2-a \\ a^2-2a \\ a \end{pmatrix}$$

1. [40%] Determine a, b and c such that the system has a unique solution.

   $1 \cdot (-b+c)a \neq 0$

2. [20%] Determine a, b and c such that the system has no solution.

   $-b+c=0$, $a \neq 0$ signal eq.

3. [20%] Determine a, b and c such that the system has infinitely many solutions.

   $a=0$

(a) There are at least 5 free variables because rank+nullity = 15 and rank ≤ 10. The general solution looks like

$$\vec{x} = t_1 \vec{x}_1 + t_2 \vec{x}_2 + \cdots + t_k \vec{x}_k$$

where $t_1, \ldots, t_k$ are invented symbols, $k \ge 5$, and $\vec{x}_1, \ldots, \vec{x}_k$ are independent vectors from the last frame algorithm. So $\vec{x}_1, \vec{x}_2$ satisfy the requirements.

(b)
$$\left(\begin{array}{ccc|c} 1 & b+c & a & a^2-a \\ 3 & 2b+4c & a & a^2-2a \\ -1 & -b-c & 0 & a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & b+c & a & a^2-a \\ 0 & -b+c & -2a & -2a^2+a \\ -1 & -b-c & 0 & a \end{array}\right) \quad \text{combo}(1,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & b+c & a & a^2-a \\ 0 & -b+c & -2a & -2a^2+a \\ 0 & 0 & a & a^2 \end{array}\right) \quad \text{combo}(1,3,1)$$

$$\left(\begin{array}{ccc|c} 1 & b+c & a & a^2-a \\ 0 & -b+c & 0 & a \\ 0 & 0 & a & a^2 \end{array}\right) \quad \text{combo}(3,2,2)$$

$$\left(\begin{array}{ccc|c} 1 & b+c & 0 & -a \\ 0 & -b+c & 0 & a \\ 0 & 0 & a & a^2 \end{array}\right) \quad \text{combo}(3,1,-1)$$

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4. (Matrix algebra)
Do both problems [50% each].

(a) [50%] Explain why there are no invertible 2×2 matrices A such that

$$A^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

(b) [50%] Display two 3×3 matrices A, B such that

$$(A^2 + AB + B^2)(A - B) \neq A^3 - B^3.$$

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ holds for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} 3a-b & a \\ 3c-d & c \end{pmatrix} \iff \begin{cases} c = 3a-b \\ d = a \\ a = 3c-d \\ b = c \end{cases} \Rightarrow \begin{cases} 3a - 2c = 0 \\ 2a - 3c = 0 \end{cases}$

$\begin{pmatrix} 3 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has unique solution $a = c = 0$ ($\det \neq 0$)

Then $a = b = c = d = 0$, so $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible.

(b) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then

$AB = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A - B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$,

$A^2 = B^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$$(A^2 + AB + B^2)(A - B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$A^3 - B^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So the two sides are unequal.

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5. (Geometry and linear transformations)

(a) [40%] Classify the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ geometrically as scaling, projection onto line L, reflection in line L, pure rotation by angle θ, rotation composed with scaling, horizontal shear, vertical shear.

1. $A = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} = \sqrt{8}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ = Scale + pure rotation

2. $A = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y-5x \end{pmatrix}$ x fixed, y changes vertical shear

3. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, $a^2+b^2=1$ Reflection

4. $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\vec{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $u_1=0$, $u_2=1$, $u_3=0$ projection onto dir $\vec{u}$.

(b) [60%] Supply all details. Figures expected.

5. Display the matrix of a rotation counter-clockwise by angle $\pi/4$ in $\mathcal{R}^2$. $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

6. Define projection onto a line L in $\mathcal{R}^3$ with direction $\vec{\mathbf{u}}$, $\|\vec{\mathbf{u}}\| = 1$.

$$\text{proj}_L(\vec{x}) = (\vec{u}\cdot\vec{x})\vec{u}$$

7. Display the matrix A of a reflection in $\mathcal{R}^3$ through the line L in the direction of vector $\vec{\mathbf{v}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. $\vec{u} = \frac{1}{\sqrt{2}}\vec{v}$ unitizes $\vec{v}$, $|\vec{u}|=1$. $u_1 = 0$, $u_2 = 1/\sqrt{2}$, $u_3 = 1/\sqrt{2}$

$$A = \begin{pmatrix} 2u_1u_1-1 & 2u_2u_1 & 2u_3u_1 \\ 2u_1u_2 & 2u_2u_2-1 & 2u_3u_2 \\ 2u_1u_3 & 2u_2u_3 & 2u_3u_3-1 \end{pmatrix}.$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Figure 5.

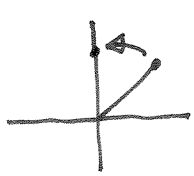


Figure 6.

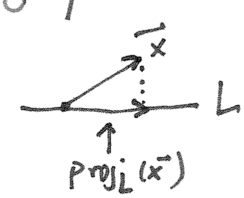
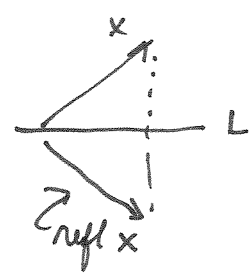


Figure 7.



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