Differential Equations and Linear Algebra

2250-1 [7:30 class] at 8:00am on 5 May 2010

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All scratch work for a chapter must appear in order. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.
- Records will be posted at the Registrar's web site on WEBct. Recording errors are reported by email.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please discard this page or keep it for your records.

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Ch3. (Linear Systems and Matrices) Complete all problems.

[10%] **Ch3(a)**: Check the correct box. Incorrect answers lose all credit.
Part 1. [5%]: ☐ True or ☐ False:
If the 7×7 matrix A is triangular and invertible, then all diagonal elements of A are nonzero.
Part 2. [5%]: ☐ True or ☐ False:
If a 3×3 matrix A has no row of zeros, then $\mathbf{rref}(A)$ is the identity matrix.

Answer: True. False.

[30%] **Ch3(b)**: Determine which values of k correspond to (1) a unique solution, (2) infinitely many solutions and (3) no solution, for the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ given by

$$A = \begin{pmatrix} 0 & k-1 & k-3 \\ 1 & 5 & k \\ 1 & 5 & 3 \end{pmatrix}, \quad \vec{\mathbf{b}} = \begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix}.$$

Answer: Elimination methods with swap, combo, multiply give $\begin{pmatrix} 1 & 5 & 3 & k \\ 0 & k-1 & 0 & k-1 \\ 0 & 0 & k-3 & 2-k \end{pmatrix}$.
Then (1) Unique solution for $k \neq 1$ and $k \neq 3$; (2) No solution for $k = 3$ [signal equation]; (3) Infinitely many solutions for $k = 1$.

[20%] **Ch3(c)**: Let matrix C and vector $\vec{\mathbf{b}}$ be defined by the equations

$$C = \begin{pmatrix} -3 & 3 & 0 \\ 0 & -3 & 4 \\ 1 & 0 & -4 \end{pmatrix}, \quad \vec{\mathbf{b}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Let I denote the 3×3 identity matrix. Find the value of x_2 by Cramer's Rule in the system $(4I + C)\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

Answer: $x_2 = \Delta_2/\Delta$, $\Delta_2 = -8$, $\Delta = \det(4I + C) = 12$, $x_2 = -2/3$.

[20%] **Ch3(d)**: Display the entry in row 4, column 2 of the adjugate matrix [or adjoint matrix] of

$$A = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Answer: answer = cofactor of A in row 2, column 4 = $(-1)^6$ times minor of A in 2, 4 = -3.

[20%] **Ch3(e)**: Assume $A = B^T$, where $B = \begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$. Symbol B^T means the transpose of B. Find the inverse of A.

Answer: $A = B^T = \begin{pmatrix} 2 & 0 \\ -6 & 4 \end{pmatrix}$, $A^{-1} = \frac{1}{8}\begin{pmatrix} 4 & 0 \\ 6 & 2 \end{pmatrix}$.

Scores
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch8.
Ch9.
Ch10.

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Ch4. (Vector Spaces) Complete all problems.

[20%] **Ch4(a)**: Define S to be the set of all vectors $\vec{\mathbf{x}}$ in $\mathcal{R}^4$ such that $x_1 + x_3 = x_2$ and $x_3 + x_4 = x_1$. Prove that S is a subspace of $\mathcal{R}^4$.

Answer: Let $A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then the restriction equations can be written as $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Apply the kernel theorem. This is theorem 2 in section 4.2 of Edwards-Penney.

[20%] **Ch4(b)**: Give an example of three vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$, each with five components, such that the nullity of their 5×3 augmented matrix is two.

Answer: Let $\vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_3$ for any nonzero vector $\vec{\mathbf{v}}_1$.

[30%] **Ch4(c)**: Apply an independence test to the vectors below. Report **independent** or **dependent**. Details count.

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 0 \end{pmatrix}.$$

Answer: Dependent. The rank of the augmented matrix of the three vectors is 2.

[30%] **Ch4(d)**: Find a basis of fixed vectors in $\mathcal{R}^4$ for the solution space of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$, where the 4×4 matrix A is given below.

$$A = \begin{pmatrix} 3 & -1 & 1 & 1 \\ 4 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

Answer: $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, basis $= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}$

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Ch5. (Linear Equations of Higher Order) Complete all problems.

[10%] **Ch5(a)**: Report the general solution $y(x)$ of the differential equation

$$5\frac{d^3y}{dx^3} + 26\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0.$$

Answer: Characteristic equation $r(5r+1)(r+5) = 0$ has roots 0, $-1/5$, -5. The general solution $y(x)$ is a linear combination of the atoms 1, $e^{-x/5}$, e^{-5x}.

[20%] **Ch5(b)**: Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 2$, $c = 1 + a$, $k = 2 + a$ and symbol $a > 0$, calculate all values of symbol a such that the solution $x(t)$ is **under-damped**. Please, **do not solve** the differential equation!

Answer: The discriminant must be negative. Then $(1+a)^2 - 8(2+a) < 0$ or $a^2 - 7a - 15 < 0$.

[20%] **Ch5(c)**: A particular solution of the differential equation $x'' + 2x' + 17x = 50\cos(3t)$ is

$$x(t) = 4\cos 3t + 12e^{-t}\sin 4t + 3\sin 3t + 15e^{-t}\cos 4t.$$

Identify the **steady-state** solution $x_{\text{SS}}(t)$ and the **transient solution** $x_{\text{tr}}(t)$.

Answer: The transient solution is the sum of all terms with limit zero. The steady-state is the sum of the remaining terms. Then $x_{ss} = 4\cos 3t + 3\sin 3t$ and $x_{\text{tr}} = 12e^{-t}\sin 4t + 15e^{-t}\cos 4t$.

[20%] **Ch5(d)**: Determine a basis of solutions of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$r(r^2 - r)^2((r+1)^2 + 11)^2 = 0.$$

Answer: The roots are $0, 0, 0, 1, 1, -1 \pm \sqrt{11}i, -1 \pm \sqrt{11}i$. By Euler's theorem, a basis is the set of atoms for these roots: $1, x, x^2, e^x, xe^x, e^{-x}\cos(\sqrt{11}x), e^{-x}\sin(\sqrt{11}x), xe^{-x}\cos(\sqrt{11}x), xe^{-x}\sin(\sqrt{11}x)$.

[30%] **Ch5(e)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} = 2x^3 + 3x\cos 2x + 4e^{2x}$$

Answer: Let $f(x) = 2x^3 + 3x\cos 2x + 4e^{2x}$. The atoms in f are x^3, $x\cos 2x$, e^{2x}. The completed set of atoms, including all lower-power related atoms, is $1, x, x^2, x^3, \cos 2x, x\cos 2x, \sin 2x, x\sin 2x, e^{2x}$. There are 9 atoms in this list. A theorem says that the shortest trial solution contains 9 atoms. Break the 9 atoms into four groups, each with the same base atom: group $1 == 1, x, x^2, x^3$; group $2 == \cos 2x, x\cos 2x$; group $3 == \sin 2x, x\sin 2x$; group $4 == e^{2x}$. Modify each group, to remove any solution of the homogeneous equation $y^{(4)} + 4y^{(2)} = 0$. Then the four groups are replaced by group $1^* == x^2, x^3, x^4, x^5$; group $2^* == x\cos 2x, x^2\cos 2x$; group $3^* == x\sin 2x, x^2\sin 2x$; group $4^* == e^{2x}$. The shortest trial solution is a linear combination of these last nine atoms.

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems.

[40%] **Ch6(a)**: Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -1 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 5 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer: The characteristic polynomial is $\det(A - rI) = (r^2+3)(5-r)(r-2)^2$. The eigenvalues are $2, 2, 5, \pm\sqrt{3}i$. Determinant expansion is by the cofactor method along column 5. This reduces it to a 4×4 determinant, which can be expanded as a product of two quadratics.

[30%] **Ch6(b)**: Find the eigenvectors corresponding to complex eigenvalues $-1 \pm 2i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$.

Answer: $\left(-1+2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right), \left(-1-2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$

[30%] **Ch6(c)**: Let $A = \begin{pmatrix} -11 & 6 \\ -18 & 10 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(-2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(-2, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

Answer: The first is not, the others yes, because the test $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ passes for the last two pairs.

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Ch7. (Linear Systems of Differential Equations) Complete all problems.

[30%] **Ch7(a)**: Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\begin{aligned}\frac{dx}{dt} &= x + y,\\ \frac{dy}{dt} &= 4x + y.\end{aligned}$$

Answer: Define $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$. The eigenvalues $-1, 3$ are roots of the characteristic equation $\det(A - rI) = (r+1)(r-3) = 0$. By Cayley-Hamilton-Zeibur, $x(t) = c_1 e^{-t} + c_2 e^{3t}$. Using the first differential equation $x' = x + y$ implies $y(t) = x' - x = -2c_1 e^{-t} + 2c_2 e^{3t}$.

[30%] **Ch7(b)**: Let A be an $n \times n$ matrix of real numbers. One method studied to solve the system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ was the eigenanalysis method, which requires finding n eigenpairs of A. State **two different methods** for solving the system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, which you learned in this course, and **briefly describe each method**.

Answer: Besides eigenanalysis, there is Zeibur-Cayley-Hamilton, Laplace resolvent, exponential matrix.

[40%] **Ch7(c)**: Define

$$A = \begin{pmatrix} -3 & 4 & -8 \\ 0 & 2 & 0 \\ 5 & -4 & 10 \end{pmatrix}$$

The eigenvalues of A are 5, 2, 2. Apply the eigenanalysis method, which requires eigenvalues and eigenvectors, to solve the differential system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$.

Answer: Form $A_1 = A - (5)I = \begin{pmatrix} -8 & 4 & -8 \\ 0 & -3 & 0 \\ 5 & -4 & 5 \end{pmatrix}$. Then $\mathbf{rref}(A_1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The last frame algorithm implies general solution of $A_1\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is $x_1 = -t_1$, $x_2 = 0$, $x_3 = t_1$. Take the partial on symbol t_1 to obtain the eigenvector $\vec{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Repeat with $A_2 = A - (2)I = \begin{pmatrix} -5 & 4 & -8 \\ 0 & 0 & 0 \\ 5 & -4 & 8 \end{pmatrix}$. Then $\mathbf{rref}(A_2) = \begin{pmatrix} 1 & -\frac{4}{5} & \frac{8}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The last frame algorithm implies general solution of $A_1\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is $x_1 = 4t_1/5 - 8t_2/5$, $x_2 = t_1$, $x_3 = t_2$. Take the partial on symbols t_1, t_2 to obtain the eigenvectors $\vec{\mathbf{v}}_2 = \begin{pmatrix} \frac{4}{5} \\ 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_3 = \begin{pmatrix} -\frac{8}{5} \\ 0 \\ 1 \end{pmatrix}$. The general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ is $\vec{\mathbf{u}}(t) = c_1 e^{5t}\vec{\mathbf{v}}_1 + c_2 e^{2t}\vec{\mathbf{v}}_2 + e^{2t}\vec{\mathbf{v}}_3$.

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Ch8. (Matrix Exponential) Complete all problems.

[30%] **Ch8(a)**: Consider the 2×2 system

$$\begin{aligned} x' &= -x, \\ y' &= 2y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Solve the system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ for $\vec{\mathbf{u}}$, using the matrix exponential e^{At}.

Answer: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ 2e^{2t} \end{pmatrix}.$

[40%] **Ch8(b)**: Display the matrix form of variation of parameters for the 2×2 system. Then integrate to find one particular solution.

$$\begin{aligned} x' &= -x + 3, \\ y' &= 2y + 1. \end{aligned}$$

Answer: $\vec{\mathbf{u}}_p(t) = e^{At} \int_0^t e^{-As} \begin{pmatrix} 2 \\ 1 \end{pmatrix} ds$. Then $e^{At} = \mathbf{diag}(e^{-t}, e^{2t})$ from the first problem, $e^{-As} = \mathbf{diag}(e^{s}, e^{-2s})$, and $\int_0^t \mathbf{diag}(e^{s}, e^{-2s}) \begin{pmatrix} 3 \\ 1 \end{pmatrix} ds = \int_0^t \mathbf{diag}(3e^{s}, e^{-2s}) ds = \begin{pmatrix} -3 + 3e^{t} \\ -\frac{1}{2}e^{-2t} + \frac{1}{2} \end{pmatrix}$.
Finally, $\vec{\mathbf{u}}_p(t) = \mathbf{diag}(e^{-t}, e^{2t}) \begin{pmatrix} -3 + 3e^{t} \\ -\frac{1}{2}e^{-2t} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 - 3e^{-t} \\ -\frac{1}{2} + \frac{1}{2}e^{2t} \end{pmatrix}$.

[30%] **Ch8(c)**: Check the correct statements.

☐ **1**. Assume A is $n \times n$. The matrix e^{-At} is the inverse of e^{At}.

☐ **2**. The matrix exponential e^{At} cannot usually be found from the general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$.

☐ **3**. A spring-mass equation $mx''(t) + cx'(t) + kx(t) = f(t)$ can be transformed into a first order system of the form $\vec{\mathbf{u}}' = B\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$ and then solved with matrix variation of parameters.

Answer: The first is true. The second is false. The third is true.

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Ch9. (Nonlinear Systems) Complete all problems.

[30%] **Ch9(a)**:
Determine whether the equilibrium $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ is stable or unstable. Then classify the equilibrium point $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ as a saddle, center, spiral or node.

$$\vec{\mathbf{u}}' = \begin{pmatrix} -5 & -2 \\ 1 & -2 \end{pmatrix} \vec{\mathbf{u}}$$

Answer: The eigenvalues of A are roots of $r^2 - 7r + 12 = (r-3)(r-4) = 0$. Then $r = 3, 4$ and the atoms are e^{3t}, e^{4t}. No trig terms eliminates the center and spiral. Finally, the atoms have limit zero at $t = -\infty$, therefore the system is stable at $t = -\infty$. That eliminates the saddle. So it must be a node. Report: **unstable node**.

[30%] **Ch9(b)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - y^2 + y + 9, \\ y' &= 2x + 2y. \end{aligned}$$

An equilibrium point is $x = -3$, $y = 3$. Compute the Jacobian matrix A of the linearized system at this equilibrium point.

Answer: The Jacobian is $J(x,y) = \begin{pmatrix} 1 & -2y+1 \\ 2 & 2 \end{pmatrix}$. Then $A = J(-3,3) = \begin{pmatrix} 1 & -5 \\ 2 & 2 \end{pmatrix}$.

[40%] **Ch9(c)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= 4x - 4y + 9 - x^2, \\ y' &= 3x - 3y. \end{aligned}$$

At equilibrium point $x = 3$, $y = 3$, the Jacobian matrix is $A = J(3,3) = \begin{pmatrix} -2 & -4 \\ 3 & -3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ for the linear system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$.

(2) Apply a theorem to classify $x = 3$, $y = 3$ as a saddle, center, spiral or node for **the nonlinear dynamical system**.

Answer: The Jacobian is $J = \begin{pmatrix} 4-2x & -4 \\ 3 & -3 \end{pmatrix}$. Then $A = J(3,3) = \begin{pmatrix} -2 & -4 \\ 3 & -3 \end{pmatrix}$. The eigenvalues of A are found from $r^2 + 5r + 17 = 0$, giving roots $a \pm bi$ where $a = -\frac{5}{2}$ and $b = \frac{1}{2}\sqrt{47}$. Because both atoms $e^{at}\cos bt$, $e^{at}\sin bt$ have trig functions, rotation happens, which eliminates the saddle and the node. The exponential e^{at} says that it is not a center, but a spiral. The atoms have limit zero at $t = \infty$, therefore we report a **stable spiral** for the linear problem $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ at equilibrium $\vec{\mathbf{u}} = \vec{\mathbf{0}}$. Theorem 2 in 9.2 applies to say that the same is true for the nonlinear system: **stable spiral** at $x = 3$, $y = 3$.

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Ch10. (Laplace Transform Methods) Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[20%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables:

$f(t)$					
$\mathcal{L}(f(t))$	$\frac{1}{s^2}$	$\frac{1}{s^{200}}$	$\frac{1}{s-a}$	$\frac{s}{s^2+b^2}$	$\frac{b}{s^2+b^2}$

$f(t)$	$1+t$	te^{2t}	$e^t\cos 2t$	$e^t(t+\sin 2t)$	$(2+t)^2$
$\mathcal{L}(f(t))$					

Answer: First table left to right: t, $\frac{t^{199}}{(199)!}$, e^{at}, $\cos bt$, $\sin bt$. Second table left to right: $\frac{1}{s}+\frac{1}{s^2}$, $\frac{1}{(s-2)^2}$, $\frac{s-1}{(s-1)^2+4}$, $\frac{1}{(s-1)^2}+\frac{2}{(s-1)^2+4}$, $\frac{2}{s}+\frac{4}{s^2}+\frac{2}{s^3}$.

[20%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for $f(t)=e^{-t}$ on $t\ge 2$, $f(t)=0$ otherwise.

Answer: Use $f(t)=e^{-t}g(t)$, $g(t)=\textbf{step}(t-2)$, and the second shifting theorem. Then $\mathcal{L}(f(t))=\mathcal{L}(e^{-t}\,\textbf{step}(t-2))=e^{-2s}\mathcal{L}(e^{-t}|_{t\to t+2})=e^{-2s}\mathcal{L}(e^{-t-2})=e^{-2s}\frac{e^{-2}}{s+1}$.

[20%] **Ch10(c)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t))=\dfrac{e^{-s}}{s^3}$.

Answer: Use the second shifting theorem, $e^{-as}\mathcal{L}(h(t))=\mathcal{L}(h(t-a)\,\textbf{step}(t-a))$. Then $\mathcal{L}(f(t))=e^{-s}\frac{1}{s^2}=e^{-s}\mathcal{L}(t^2/2)=\frac{1}{2}\mathcal{L}(\,(t^2)|_{t\to t-1})=\frac{1}{2}\mathcal{L}((t-1)^2\,\textbf{step}(t-1))$. Lerch's theorem implies $f(t)=\frac{1}{2}(t-1)^2\,\textbf{step}(t-1)$.

[20%] **Ch10(d)**: Solve for $f(t)$ in the equation $\dfrac{d}{ds}\mathcal{L}(f(t))=\dfrac{2}{(s+1)^2}+\dfrac{d^2}{ds^2}\mathcal{L}(e^t\cos t)$.

Answer: $\mathcal{L}((-t)f(t))=\frac{2}{u^2}\Big|_{u=s+1}+\mathcal{L}((-t)^2e^t\cos t)=\mathcal{L}(e^{-t}(2t)+t^2e^t\cos t)$, giving the final answer $f(t)=-2e^{-t}-te^t\cos t$. The Laplace details use the first shifting theorem and s-differentiation theorem.

[20%] **Ch10(e)**: Solve by Laplace's method for the solution $x(t)$:

$$x''(t) - x'(t) = e^t, \quad x(0) = x'(0) = 0.$$

Answer: $x(t) = (t-1)e^{-t} + 1$. The Laplace steps are: (1) $\mathcal{L}(x(t)) = \dfrac{1}{s(s-1)^2}$; (2) $\mathcal{L}(x(t)) = \dfrac{1}{s(s-1)^2} = \dfrac{A}{s} + \dfrac{B}{s-1} + \dfrac{C}{(s-1)^2} = \mathcal{L}(A + Be^t + Cte^t)$ where by partial fractions the answers are $A = 1$, $B = -1$, $C = 1$. Lerch's theorem implies $x(t) = 1 - e^{-t} + te^{-t}$.

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