Differential Equations and Linear Algebra

2250-4 [12:55 class] at 1:00pm on 3 May 2010

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All scratch work for a chapter must appear in order. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.
- Records will be posted at the Registrar's web site on WEBct. Recording errors are reported by email.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please discard this page or keep it for your records.

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Differential Equations and Linear Algebra 2250-4 [12:55 class]

Final Exam at 1:00pm on 3 May 2010

Ch3. (Linear Systems and Matrices) Complete all problems.

[10%] **Ch3(a)**: Check the correct box. Incorrect answers lose all credit.

Part 1. [5%]: ☑ True or ☐ False:

If the 10×10 matrices A and B are lower triangular, then AB is triangular.

Part 2. [5%]: ☐ True or ☑ False:

If a 3×3 matrix A has a row of zeros, then for all vectors $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions $\mathbf{x}$.

[30%] **Ch3(b)**: Determine which values of k correspond to (1) a unique solution, (2) infinitely many solutions and (3) no solution, for the system $A\mathbf{x} = \mathbf{b}$ given by

$$A = \begin{pmatrix} 0 & k-2 & k-3 \\ 1 & 4 & k \\ 1 & 4 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ k \end{pmatrix}.$$

Unique: $(k-2)(k-3) \neq 0$
∞-many: $k=2$
No Sol: $k=3$ signal Eq.

[20%] **Ch3(c)**: Let matrix C and vector $\mathbf{b}$ be defined by the equations

$$C = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$x_2 = -\frac{2}{3}$

Let I denote the 3×3 identity matrix. Find the value of x_2 by Cramer's Rule in the system $(3I+C)\mathbf{x} = \mathbf{b}$.

[20%] **Ch3(d)**: Display the entry in row 3, column 4 of the adjugate matrix [or adjoint matrix] of

$$A = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

entry $= -2$

[20%] **Ch3(e)**: Assume $A^{-1} = \begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$. Find the transpose of A.

(b) $\left(\begin{array}{ccc|c} 1 & 4 & 3 & k \\ 0 & k-2 & 0 & k-2 \\ 0 & 0 & k-3 & 1-k \end{array}\right)$ = Frame 4, Triangular

(c) $3I+C = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \end{pmatrix}$, $x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 3 & 0 \end{vmatrix}}{12} = \dfrac{-8}{12}$

(d) Entry $=$ cofactor$(A,4,3) = (-1)^{4+3}$ minor$(A,4,3)$

$$= -\begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix} = -2$$

(e) $A = \dfrac{\text{adj}(A^{-1})}{|A^{-1}|} = \dfrac{1}{8}\begin{pmatrix} 4 & 6 \\ 0 & 2 \end{pmatrix} \Rightarrow A^T = \dfrac{1}{8}\begin{pmatrix} 4 & 0 \\ 6 & 2 \end{pmatrix}$

Scores
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch8.
Ch9.
Ch10.

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Differential Equations and Linear Algebra 2250-4 [12:55 class]

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Ch4. (Vector Spaces) Complete all problems.

[20%] **Ch4(a)**: Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ such that $x_1 + x_3 = 0$ and $x_3 + x_4 = x_1$. Prove that S is a subspace of $\mathcal{R}^4$.

[20%] **Ch4(b)**: Give an example of three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ for which the nullity of their augmented matrix is two.

[30%] **Ch4(c)**: Apply an independence test to the vectors below. Report **independent** or **dependent**. Details count.

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

[30%] **Ch4(d)**: Find a basis of fixed vectors in $\mathcal{R}^4$ for the solution space of $A\mathbf{x} = \mathbf{0}$, where the 4×4 matrix A is given below.

$$A = \begin{pmatrix} 3 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

(a) Define $B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then the equations are equivalent to $B\vec{x} = \vec{0}$. Apply the Kernel Theorem, Thm 2 in 4.2 of Edwards-Penney, S is a subspace.

(b) Let v_1, v_2, v_3 be the columns of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\begin{cases} \text{rank} = 1 \\ \text{nullity} = 2 \end{cases}$

(c) rref (augmented matrix of v_1, v_2, v_3) $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Independent by the rank test.

(d) $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{cases} x_1 = -t_2/2 \\ x_2 = t_1 - t_2/2 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$ $\quad$ Basis $= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}$

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Differential Equations and Linear Algebra 2250-4 [12:55 class]

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Ch5. (Linear Equations of Higher Order) Complete all problems.

[10%] **Ch5(a)**: Report the general solution $y(x)$ of the differential equation

$$3\frac{d^3y}{dx^3} + 10\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0.$$

[20%] **Ch5(b)**: Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 2$, $c = 2 + a$, $k = 1 + a$ and $a > 0$ a symbol, calculate all values of symbol a such that the solution $x(t)$ is **over-damped**. Please, **do not solve** the differential equation!

[20%] **Ch5(c)**: A particular solution of the differential equation $x'' + 2x' + 17x = 50\cos(3t)$ is

$$x(t) = 4\cos 3t + 12e^{-t}\sin 4t + 3\sin 3t + 15e^{-t}\cos 4t.$$

Identify the **steady-state** solution $x_{SS}(t)$.

[20%] **Ch5(d)**: Determine a basis of solutions of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$r(r^2 + r)^2((r + 1)^2 + 7)^2 = 0.$$

[30%] **Ch5(e)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4y}{dx^4} + 9\frac{d^2y}{dx^2} = 2x^2 + 3x\sin 3x + 4e^x$$

(a) Char eq: $r(3r+1)(r+3)=0$. Then $y =$ linear combination of atoms, which are $1, e^{-x/3}, e^{-3x}$

(b) The discriminant, the expression under $\sqrt{\ }$ in the quadratic formula, must be positive. Then $0 < c^2 - 4km = a^2 - 4a - 4$.

(c) $x_{SS} = 4\cos 3t + 3\sin 3t$, because the other terms limit to zero at ∞.

(d) A basis is the list of 9 atoms: $r = 0,0,0,-1,-1,-1\pm\sqrt{7}i,-1\pm\sqrt{7}i$

$1, x, x^2, e^{-x}, xe^{-x}, e^{-x}\cos\sqrt{7}x, xe^{-x}\cos\sqrt{7}x, e^{-x}\sin\sqrt{7}x, xe^{-x}\sin\sqrt{7}x$

(e)

		Corrected list
group 1 :	$1, x, x^2$	x^2, x^3, x^4
group 2 :	$\cos 3x, x\cos 3x$	$x\cos 3x, x^2\cos 3x$
group 3 :	$\sin 3x, x\sin 3x$	$x\sin 3x, x^2\sin 3x$
group 4 :	e^x	e^x

trial solution = linear combination of 8 corrected atoms on the left.

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Char eq: $r^4 + 9r^2 = 0$, $r = 0, 0, \pm 3i$

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Differential Equations and Linear Algebra 2250-4 [12:55 class]

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems.

[40%] **Ch6(a)**: Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -2 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

[30%] **Ch6(b)**: Find the eigenvectors corresponding to complex eigenvalues $-1 \pm 3i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$.

[30%] **Ch6(c)**: Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

(a) $|A - rI| = (3-r)\begin{vmatrix} B_1 & B_2 \\ 0 & B_3 \end{vmatrix} = (3-r)\,|B_1|\,|B_3| = (3-r)(r^2+6)(r^2-4r+4)$

Cofactor expansion along column 5

Block matrix theorem

Eigenvalues $= 3, \pm\sqrt{6}i, 2, 2$

(b) $A - (-1+3i)I = \begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix} \to \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \to \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \to \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$

$\begin{cases} x_1 = -i t_1 \\ x_2 = t_1 \end{cases} \Rightarrow \vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$. Theorem $\Rightarrow$ eigenpairs $\left(-1+3i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right), \left(-1-3i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$

(c) $\begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -10 & -2 \\ 2 & -17 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ are eigenvectors

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Differential Equations and Linear Algebra 2250-4 [12:55 class]

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Ch7. (Linear Systems of Differential Equations) Complete all problems.

[30%] **Ch7(a)**: Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = 6x + 2y.$$

$$\begin{cases} x = c_1 e^{4t} + c_2 e^{-t} \\ y = 3c_1 e^{4t} - 2c_2 e^{-t} \end{cases}$$

[30%] **Ch7(b)**: Let A be an $n \times n$ matrix of real numbers. State three different methods for solving the system $\vec{u}' = A\vec{u}$, which you learned in this course.

[40%] **Ch7(c)**: Define

$$A = \begin{pmatrix} -3 & 4 & -10 \\ 0 & 2 & 0 \\ 5 & -4 & 12 \end{pmatrix}$$

$\lambda = 7,\ \vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\lambda_2 = 2,\ \vec{v}_2 = \begin{pmatrix} 4/5 \\ 1 \\ 0 \end{pmatrix}$, $\lambda_3 = 2,\ \vec{v}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

The eigenvalues of A are 7, 2, 2. Apply the eigenanalysis method, which requires eigenvalues and eigenvectors, to solve the differential system $\mathbf{u}' = A\mathbf{u}$.

(a) Zeibur shortcut. x = linear combination of atoms

$r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0 \Rightarrow r = 4, -1 \Rightarrow$ atoms e^{4t}, e^{-t}

$x = c_1 e^{4t} + c_2 e^{-t}$. $y = x' - x = 3c_1 e^{4t} - 2c_2 e^{-t}$

(b) Zeibur-Cayley-Hamilton, $\vec{u}$ = vector linear combination of the atoms found from $|A - rI| = 0$.

Eigenanalysis. $\vec{u} = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$ when A has n eigenpairs

Laplace Resolvent: $\vec{u}$ found from $(sI - A)\mathcal{L}(\vec{u}) = \vec{u}(0)$.

Exponential: $\vec{u} = e^{At}\vec{u}(0)$.

(c) $A - 7I = \begin{pmatrix} -10 & 4 & -10 \\ 0 & -5 & 0 \\ 5 & -4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & -4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} x_1 = -t_1 \\ x_2 = 0 \\ x_3 = t_1 \end{cases}$ $\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$A - 2I = \begin{pmatrix} -5 & 4 & -10 \\ 0 & 0 & 0 \\ 5 & -4 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} -5 & 4 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{4}{5} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} x_1 = \frac{4}{5}t_1 - 2t_2 \\ x_2 = t_1 \\ x_3 = t_2 \end{cases}$ $\vec{v}_2 = \begin{pmatrix} 4/5 \\ 1 \\ 0 \end{pmatrix}$

$\vec{v}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

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Differential Equations and Linear Algebra 2250-4 [12:55 class]

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Ch8. (Matrix Exponential) Complete all problems.

[30%] **Ch8(a)**: Consider the 2×2 system

$$x' = x,\quad y' = -2y,\quad x(0) = 1,\quad y(0) = 2.$$

$$\begin{cases} x = e^{t} \\ y = 2e^{-2t} \end{cases}$$

Solve the system $\mathbf{u}' = A\mathbf{u}$ for $\mathbf{u}$, using the matrix exponential e^{At}.

[40%] **Ch8(b)**: Display the matrix form of variation of parameters for the 2×2 system. Then integrate to find one particular solution.

$$x' = x + 3,\quad y' = -2y + 1.$$

$$\begin{cases} x_p = 3e^{t} - 3 \\ y_p = \frac{1}{2} - \frac{1}{2}e^{-2t} \end{cases}$$

[30%] **Ch8(c)**: Check the correct statements.

- [ ] 1. The system $\mathbf{u}' = A\mathbf{u}$ can only be solved when A is diagonalizable. False, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{u} = \begin{pmatrix} c_1 t + c_2 \\ c_1 \end{pmatrix}$
- [x] 2. The matrix exponential e^{At} can be found using Laplace theory. $\mathcal{L}(e^{At}) = (sI - A)^{-1}$
- [x] 3. A second order system $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}} + \vec{\mathbf{G}}(t)$ can be transformed into a first order system of the form $\vec{\mathbf{u}}' = B\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$. Yes, see Edwards-Penney Chapter 7

(a) $\begin{pmatrix} x \\ y \end{pmatrix} = e^{At}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{t} \\ 2e^{-2t} \end{pmatrix}$. Theorem. $e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}t} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

(b) use $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, same as Ch8(a)

$$u_p(t) = e^{At}\int_0^t e^{-Ax}\vec{F}(x)\,dx$$

$$= e^{At}\int_0^t \begin{pmatrix} e^{-x} & 0 \\ 0 & e^{2x} \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}dx$$

$$= e^{At}\begin{pmatrix} \int_0^t 3e^{-x}dx \\ \int_0^t e^{2x}dx \end{pmatrix}$$

$$= \begin{pmatrix} e^{t}\int_0^t 3e^{-x}dx \\ e^{-2t}\int_0^t e^{2x}dx \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{t} - 3 \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{pmatrix}$$

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Differential Equations and Linear Algebra 2250-4 [12:55 class]

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Ch9. (Nonlinear Systems) Complete all problems.

[30%] **Ch9(a):**
Determine whether the equilibrium $\mathbf{u} = \mathbf{0}$ is stable or unstable. Then classify the equilibrium point $\mathbf{u} = \mathbf{0}$ as a saddle, center, spiral or node.

$$\mathbf{u}' = \begin{pmatrix} 5 & 10 \\ -4 & -7 \end{pmatrix} \mathbf{u}$$

$r^2 + 2r + 5 = (r+1)^2 + 4$
$r = -1 \pm 2i$
Stable spiral, because rotation plus exponential.

[30%] **Ch9(b):** Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - y^2 + y + 9, \\ y' &= 2x + 2y. \end{aligned}$$

$J(3,-3) = \begin{pmatrix} 1 & -2y+1 \\ 2 & 2 \end{pmatrix} \Big|_{\substack{x=3 \\ y=-3}}$

An equilibrium point is $x = 3$, $y = -3$. Compute the Jacobian matrix A of the linearized system at this equilibrium point.

ans: $A = \begin{pmatrix} 1 & 7 \\ 2 & 2 \end{pmatrix}$

[40%] **Ch9(c):** Consider the nonlinear dynamical system

$$\begin{aligned} x' &= 4x - 4y + 9 - x^2, \\ y' &= 3x - 3y. \end{aligned}$$

At equilibrium point $x = -3$, $y = -3$, the Jacobian matrix is $A = \begin{pmatrix} 10 & -4 \\ 3 & -3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\mathbf{u} = \mathbf{0}$ for the linear system $\mathbf{u}' = A\mathbf{u}$.

(2) Apply a theorem to classify $x = -3$, $y = -3$ as a saddle, center, spiral or node for **the nonlinear dynamical system.**

(1) $r^2 - 7r - 18 = 0 \Rightarrow (r-9)(r+2) = 0 \Rightarrow r = 9, -2$, No rotation.
unstable saddle

(2) saddle, because all but node and center transfer the picture.

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Differential Equations and Linear Algebra 2250-4 [12:55 class]

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Ch10. (Laplace Transform Methods) Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[20%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables:

$f(t)$	1	$\frac{t^{n-1}}{(n-1)!}$	e^{at}	$\cos bt$	$\sin bt$
$\mathcal{L}(f(t))$	$\frac{1}{s}$	$\frac{1}{s^n}$	$\frac{1}{s-a}$	$\frac{s}{s^2+b^2}$	$\frac{b}{s^2+b^2}$

$f(t)$	t^2	te^{at}	$e^t\cos 2t$	$e^t(t+\sin 2t)$	$(1+2t)^2$
$\mathcal{L}(f(t))$	$\frac{2}{s^3}$	$\frac{1}{(s-a)^2}$	$\frac{s-1}{(s-1)^2+4}$	$\frac{1}{(s-1)^2}+\frac{2}{(s-1)^2+4}$	$\frac{1}{s}+\frac{4}{s^2}+\frac{8}{s^3}$

[20%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for $f(t)=e^t$ on $t\ge 2$, $f(t)=0$ otherwise.

[20%] **Ch10(c)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t))=\frac{e^{-s}}{s-2}$.

[20%] **Ch10(d)**: Solve for $f(t)$ in the equation $\frac{d}{ds}\mathcal{L}(f(t))=\frac{1}{(s+1)^2}+\frac{d^2}{ds^2}\mathcal{L}(\sin t)$.

[20%] **Ch10(e)**: Solve by Laplace's method for the solution $x(t)$:

$$x''(t)+x'(t)=e^{-t},\quad x(0)=x'(0)=0.$$

(b) $\mathcal{L}(f)=\mathcal{L}(e^t u(t-2)) = e^{-2s}\mathcal{L}(e^t|_{t\to t+2}) = e^{-2s}\frac{e}{s-1}$

(c) $\mathcal{L}(f)=e^{-s}\mathcal{L}(e^{2t}) = \mathcal{L}(e^{2t}|_{t\to t-1}\, u(t-1)) \Rightarrow f(t)=e^{2t-2}u(t-1)$

(d) $\mathcal{L}(-t f(t)) = \mathcal{L}(te^{-t}) + \mathcal{L}((-t)^2\sin t) \Rightarrow f(t) = -e^{-t} - t\sin t$

(e) $(s^2+s)\mathcal{L}(x) = \frac{1}{s+1} \Rightarrow \mathcal{L}(x) = \frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$

by partial fractions. Then $\mathcal{L}(x) = \mathcal{L}(1-e^{-t}-te^{-t}) \Rightarrow$

$x(t) = 1-e^{-t}-te^{-t}$.

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