Applied Differential Equations 2250

Exam date: Wednesday, 21 April, 2010

Instructions: This in-class exam is 50 minutes. Up to 60 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Chapter 5)** Complete all.

(1a) [40%] Write the solution $x(t)$ of

$$x''(t) + 9x(t) = 60\sin(2t), \quad x(0) = x'(0) = 0,$$

as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form**.

Answer:
$x(t) = -8\sin(3t) + 12\sin(2t)$

(1b) [30%] Determine the practical resonance frequency ω for the electric current equation

$$2I'' + 7I' + 50I = 100\omega\cos(\omega t).$$

Answer:
$\omega = 1/\sqrt{LC} = 1/\sqrt{2/50} = \sqrt{25} = 5.$

(1c) [30%] Use the variation of parameters formula (33) in Edwards-Penney,

$$y_p(x) = y_1(x)\left(\int \frac{-y_2(x)f(x)}{W(x)}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(x)}dx\right)$$

to display an explicit integral formula for a particular solution y_p of the equation

$$y''(x) + 2y'(x) + 17y(x) = e^{x^2}.$$

To save time, don't try to evaluate integrals (it's impossible).

Answer:
$y_p(t) = y_1(x)\int k1(x)f(x)dx + y_2(x)\int k2(x)f(x)dx$, $f(x) = e^{x^2}$, $y_1(x) = e^{-x}\cos(4x)$, $y_2(x) = e^{-x}\sin(4x)$, $k_1(x) = -y_2(x)/W(x)$, $k_2(x) = y_1(x)/W(x)$, $W(x) = 4e^{-2x}$.

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2. **(Chapter 5)** Complete all.

(2a) [60%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 8 with roots $0, 0,\ 2, 2,\ 2i, -2i,\ 5,\ 5$ listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 2e^x + 3e^{2x} + 4x^2 + 5\sin 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The atoms of $f(x)$ are e^x, e^{2x}, x^2, $\sin 2x$. Complete the list to 7 atoms, by adding the related atoms, to obtain 5 groups, each group having exactly one base atom: (1) e^x, (2) e^{2x}, (3) $1, x, x^2$, (4) $\cos 2x$, (5) $\sin 2x$. The trial solution is a linear combination of 7 atoms, modified by rules to the new list (1) e^x, (2) x^2e^{2x}, (3) x^2, x^3, x^4, (4) $x\cos 2x$, (5) $x\sin 2x$. The root 5 of the homogeneous equation is not used in the calculation.

(2b) [40%] Let $f(x) = 4e^x + \sin^2 2x$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

Answer:

Because $\sin^2(2x) = (1 - \cos 4x)/2$ [identity $\cos 2\theta = 1 - 2\sin^2\theta$], then root $r = 0$ and roots $r = \pm 4i$ are used to produce the second term $\sin^2 2x$. Total roots: $1, 0, \pm 4i$ with product of the factors $(r-1)(r-0)(r^2+16) = r^4 - r^3 + 16r^2 - 16r$. The DE: $y^{(4)} - y''' + 16y'' - 16y' = 0$.

3. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [50%] Display the details of Laplace's method to solve the system for $x(t)$. Don't waste time solving for $y(t)$!

$$x' = 2x + 3y,$$
$$y' = x,$$
$$x(0) = 4, \quad y(0) = 0.$$

Answer:

The Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ can be written out to find a 2×2 linear system for unknowns $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$. Cramer's rule applies to this system to solve for $\mathcal{L}(x(t)) = \dfrac{4s}{(s-3)(s+1)}$. Then partial fractions and backward table methods determine $x(t) = 3e^{3t} + e^{-t}$.

(3b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{4s + 24}{s^2(s+4)}.$$

Answer:

$\mathcal{L}(f(t)) = \frac{6}{s^2} + \frac{-1/2}{s} + \frac{1/2}{s+4} = \mathcal{L}(-1/2 + 6t + (1/2)e^{-4t})$

(3c) [25%] Solve for $f(t)$, given

$$\frac{d^2}{ds^2}\mathcal{L}(f(t)) = \frac{6}{s^4}.$$

Answer:

Use the s-differentiation theorem and the backward Laplace table to get $(-t)^2 f(t) = t^3$ or $f(t) = t$.

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4. **(Chapter 10)** Complete all parts.

(4a) [50%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$e^{-t}\sin 2t$	$t^3 e^{-t/2}$
$\mathcal{L}(f(t))$	$\dfrac{6}{s^4}$	$\dfrac{1}{(3s-2)^3}$	$\dfrac{s+1}{s^2+2s+17}$		

Answer:
Left to right: $\dfrac{1}{54}t^2e^{2t/3}$, $e^{-t}\cos 4t$, $\dfrac{2}{(s+1)^2+4}$, $\dfrac{96}{(1+2s)^4}$.

(4b) [50%] Find $\mathcal{L}(f(t))$, given $f(t) = u(t-\pi)\dfrac{\sin(t)}{e^t}$, where u is the unit step function.

Answer:
Remove the factor e^{-t}, to be inserted later. Use the second shifting theorem $\mathcal{L}(u(t-a)g(t)) = e^{-as}\mathcal{L}(g(t+a))$. Then $\mathcal{L}(\sin(t)u(t-\pi)) = e^{-\pi s}\mathcal{L}\left((\sin(t))|_{t\to t-\pi}\right) = e^{-\pi s}\mathcal{L}(\sin(t-\pi)) = e^{-\pi s}\mathcal{L}(-\sin(t)) = \dfrac{-e^{-\pi s}}{s^2+1}$. When the factor e^{-t} is inserted, then the first shifting theorem applies to shift the final answer $s \to s+1$. Then $\mathcal{L}(f(t)) = \left(e^{-\pi s}\dfrac{-1}{s^2+1}\right)\Big|_{s\to s+1} = \dfrac{-e^{-\pi(s+1)}}{(s+1)^2+1}$.

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5. **(Chapter 6)** Complete all parts.

(5a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 4 & 1 & 12 \\ -4 & 1 & -3 & 15 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & -2 & 7 \end{pmatrix}$. To save time, **do not** find eigenvectors!

Answer:
$1, 5, 1 \pm 4i$

(5b) [30%] Given $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$, which has eigenvalues $1, 2, 2$, find all eigenvectors for eigenvalue 2.

Answer:
One frame sequence is required for $\lambda = 2$. The sequence starts with $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$, the last frame having two rows of zeros. There are two invented symbols t_1, t_2 in the last frame algorithm answer $x_1 = -t_1 + t_2$, $x_2 = t_1$, $x_3 = t_2$. Taking ∂_{t_1} and ∂_{t_2} gives two eigenvectors, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(5c) [20%] Suppose a 3×3 matrix A has eigenpairs

$$\left(2, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer:
Define $P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $AP = PD$.

(5d) [20%] Assume the vector general solution $\vec{\mathbf{u}}(t)$ of the 2×2 linear differential system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ is given by

$$\vec{\mathbf{u}}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find the matrix C.

Answer:
The eigenvalues come from the exponents in the exponentials, 2 and 2. The eigenpairs are $\left(2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$, $\left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Solve $CP = PD$ to find

$C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The usual eigenpairs for C are the columns of the identity. But the eigenvalues are equal, therefore any linear combination of the two eigenvectors is also an eigenvector. This justifies the correctness of the strange eigenpairs given in the problem.

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