Name KEY

Scores
1.
2.
3.
4.
5.

2250 Midterm Exam 2, Ver 2

Thursday, 18 March, 2010

Instructions: This in-class exam is 50 minutes. Up to 30 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(The 3 Possibilities with Symbols)**
Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & b+a & c \\ 3 & 2b+4a & c \\ -1 & -b-a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c^2-c \\ c^2-2c \\ c \end{pmatrix}$$

(a) [40%] Determine a, b and c such that the system has a unique solution. $c(-b+a) \neq 0$

(b) [30%] Determine a, b and c such that the system has no solution. $c \neq 0,\ -b+a=0$

(c) [30%] Determine a, b and c such that the system has infinitely many solutions. $c=0$

$$\left(\begin{array}{ccc|c} 1 & b+a & c & c^2-c \\ 3 & 2b+4a & c & c^2-2c \\ -1 & -b-a & 0 & c \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & b+a & c & c^2-c \\ 0 & -b+a & -2c & -2c^2+c \\ -1 & -b-a & 0 & c \end{array}\right) \quad \text{combo}(1,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & b+a & c & c^2-c \\ 0 & -b+a & -2c & -2c^2+c \\ 0 & 0 & c & c^2 \end{array}\right) \quad \text{combo}(1,3,1)$$

If $-b+a \neq 0$, c arbitrary $\neq 0$, Then unique solution, due to 3 lead variables

If $-b+a \neq 0$, $c=0$, Then 2 lead variables and ∞-many

analysis of case $-b+a=0$

$$\left(\begin{array}{ccc|c} 1 & 2a & c & c^2-c \\ 0 & 0 & -2c & -2c^2+c \\ 0 & 0 & c & c^2 \end{array}\right) \quad \text{when } -b+a=0$$

$$\left(\begin{array}{ccc|c} 1 & 2a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{when } -b+a=0 \text{ and } c=0$$

∞-many solutions

$$\left(\begin{array}{ccc|c} 1 & 2a & c & c^2-c \\ 0 & 0 & -2 & -2c+1 \\ 0 & 0 & c & c^2 \end{array}\right) \quad \text{when } -b+a=0 \text{ and } c \neq 0 \quad \text{mult}(2, 1/c)$$

$$\left(\begin{array}{ccc|c} 1 & 2a & c & c^2-c \\ 0 & 0 & -2 & -2c+1 \\ 0 & 0 & 1 & c \end{array}\right) \quad \text{mult}(3, 1/c)$$

$$\left(\begin{array}{ccc|c} 1 & 2a & c & c^2-c \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & c \end{array}\right) \quad \text{combo}(3,2,2)$$

Signal equation

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2. **(Vector Spaces)** Do all parts. Details not required (a)-(d).

(a) [10%] True or false: Every subspace S of $\mathcal{R}^3$ contains the zero vector.

(b) [10%] True or false: The set of solutions $\mathbf{x}$ in $\mathcal{R}^3$ of any matrix equation $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathcal{R}^3$. If $\vec{b} \neq \vec{0}$, Then $\vec{0}$ is not in The set.

(c) [10%] True or false: Equations $x + y = 0$, $y + z = 0$ define a subspace in $\mathcal{R}^3$. Kernel Theorem

(d) [10%] True or false: Relations $x \geq 0$, $2y + z = 0$ define a subspace in $\mathcal{R}^3$. Not a Subspace Theorem

(e) [10%] Assume V is a vector space of functions. State one theorem, without proof, that concludes that a subset S of V is a subspace of V. If $S = \text{span}(\vec{v}_1, .., \vec{v}_k)$, Then S is a subspace of V.

(f) [50%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} 2x_1 & + & 10x_2 & - & 3x_3 & + & 2x_4 & = & 0, \\ x_1 & + & 4x_2 & - & 2x_3 & + & x_4 & = & 0, \\ 2x_1 & + & 12x_2 & - & 2x_3 & + & 2x_4 & = & 0, \\ & & 4x_2 & + & 2x_3 & & & = & 0. \end{array}$$

$$\left(\begin{array}{cccc|c} 2 & 10 & -3 & 2 & 0 \\ 1 & 4 & -2 & 1 & 0 \\ 2 & 12 & -2 & 2 & 0 \\ 0 & 4 & 2 & 0 & 0 \end{array}\right) \text{ has rref } = \left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & 0 \\ 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The last frame algorithm $\Rightarrow$ $\begin{cases} x_1 = 4t_1 - t_2 \\ x_2 = -\frac{1}{2}t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$

Take partials $\partial_{t_1}, \partial_{t_2}$ to find The basis $= \begin{pmatrix} 4 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Answer check: Insert each basis vector answer into The equations, to see if They work.

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3. **(Independence and Dependence)** Do **all parts.**

(a) [10%] State an independence test for three fixed vectors.

Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent $\Longleftrightarrow$ Their augmented matrix has rank $= 3$.

(b) [40%] Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ denote the columns of the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 5 & 1 \\ 1 & 3 & 3 & 0 \\ -2 & -6 & -6 & 0 \end{pmatrix}.$$

Display the details of an independence or dependence test for the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ and report the result.

(c) [40%] Extract from the list below a largest set of independent vectors.

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \\ 9 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ -1 \\ 5 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}.$$

(d) [10%] Extract from the list in (c) above a largest set of dependent vectors.

(b) $\begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 5 & 1 \\ 1 & 3 & 3 & 0 \\ -2 & -6 & -6 & 0 \end{pmatrix}$ has rref $= \begin{pmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Because rank $= 2$, Then they are dependent.

(c) $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 3 & -1 & 2 \\ 0 & 2 & 1 & 9 & 5 & 2 \end{pmatrix}$ has rref $= \begin{pmatrix} 0 & 1 & \frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Then Columns 2, 4 of A are the largest independent set, by The pivot Theorem. ans $= \vec{v}_2, \vec{v}_4$

(d) Ans $= \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6$

The pivot Theorem says $\vec{v}_1, \vec{v}_3, \vec{v}_5, \vec{v}_6$ are linear combinations of $\vec{v}_2, \vec{v}_4$. Also, $\vec{0}$ is in the list, which makes the full list dependent. A dependency relation is

$$(1)\vec{v}_1 + (0)\vec{v}_2 + (0)\vec{v}_3 + (0)\vec{v}_4 + (0)\vec{v}_5 + 0\vec{v}_6 = \vec{0}$$

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4. **(Determinants)** Do all parts.

(a) [20%] State the determinant product rule. If C, D are $n\times n$ matrices, Then $|CD| = |C||D|$.

(b) [20%] Assume given 3×3 matrices A, B. Suppose $E_3B = E_2E_1A$ and E_1, E_2, E_3 are elementary matrices representing respectively a combination, a swap, and a multiply by $-1/8$. Assume $\det(A) = 2$. Find $\det(B)$.

(c) [20%] Determine all values of x for which B^{-1} fails to exist, where B is the transpose of the matrix

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 3x & 3x+2 & 10 \\ 1 & 0 & 3x+7 \end{pmatrix}.$$

(d) [40%] Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} -1 & 0 & -1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(b) By The Theory, $|E_1| = 1$, $|E_2| = -1$, $|E_3| = \frac{-1}{8}$. Given $|A| = 2$, Then

$|E_3||B| = |E_2||E_1||A| \Rightarrow -\frac{1}{8}|B| = (-1)(1)(2) \Rightarrow \boxed{|B| = 16}$

(c) B^{-1} fails to exist $\Leftrightarrow |B| = 0$. But $|B| = |A^T| = |A|$, so it suffices to determine when $|A| = 0$.

$|A| = (3x+2)\begin{vmatrix} 2 & 2 \\ 1 & 3x+7 \end{vmatrix}$ by cofactor expansion on column 2.

$= (3x+2)(6x+12) = 6(3x+2)(x+2)$

ans: $(3x+2)(x+2) = 0$ or $\boxed{x = \frac{-2}{3}, x = -2}$

(d) entry 3,4 of $A^{-1} = \dfrac{\text{cof}(A,4,3)}{|A|}$

$= \dfrac{(-1)\,\text{minor}(A,4,3)}{|A|}$

$= \dfrac{(-1)\begin{vmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{|A|} = \boxed{-2}$

$|A| = -1$; $|A|$ will be computed by cofactor expansion on Column 3:

$|A| = \begin{vmatrix} -1 & 0 & -1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} + (1)\begin{vmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = (-1)(-1) + (1)(-2) = -1$

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5. **(Linear Differential Equations)** Do all parts.

(a) [20%] Solve for the general solution of $y'' + 4y' + 68y = 0$.

(b) [40%] The characteristic equation is $r^3(r+2)^2(r^2+2r+10) = 0$. Find the general solution y of the linear homogeneous constant-coefficient differential equation.

(c) [20%] A second order linear homogeneous differential equation with real constant coefficients has a solution $e^{2x}\cos x$. What are the roots of the characteristic equation?

(d) [20%] **Circle the functions** which can be a solution of a linear homogeneous differential equation with constant coefficients. For example, you would circle $\cos^2 x$ because $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ is a linear combination of the two solutions 1 and $\cos(2x)$ of the third order equation whose characteristic equation has roots $0,\ 2i,\ -2i$.

$e^{x/e}$ 100 e^{x^2} $\cos(\ln|x|)$ $\cos(3x)$

$\tan(3x)$ $1/e$ $\cosh(x)$ $\sin^2(x)$ $\sin(x^2)$

(a) $r^2+4r+68=0 \Rightarrow (r+2)^2+64=0 \Rightarrow r=-2\pm 8i$

atoms $= e^{-2x}\cos(8x),\ e^{-2x}\sin(8x)$ y = linear combination of the atoms

(b) roots $r = 0,0,0,-2,-2,\ -1\pm 3i$

atoms $e^{0x},\ xe^{0x},\ x^2e^{0x},\ e^{-2x},\ xe^{-2x},\ e^{-x}\cos(3x),\ e^{-x}\sin(3x)$

y = linear combination of seven (7) atoms

(c) $e^{2x}\cos(x)$ comes from root $= 2+i$ in Euler's Theorem 3.
Because the conjugate is also a root, then $\boxed{\text{roots} = 2\pm i}$

(d) circled above.

$\cosh(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$

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