2250 Midterm Exam 2, Ver 1
Wednesday, 17 March, 2010

Instructions: This in-class exam is 50 minutes. Up to 30 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (The 3 Possibilities with Symbols)
   Let $a$, $b$ and $c$ denote constants and consider the system of equations
   \[
   \begin{pmatrix}
   1 & b + c & a \\
   3 & 2b + 4c & a \\
   -1 & -b - c & 0 \\
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y \\
   z \\
   \end{pmatrix}
   =
   \begin{pmatrix}
   a^2 - a \\
   a^2 - 2a \\
   a \\
   \end{pmatrix}
   \]

   (a) [40%] Determine $a$, $b$ and $c$ such that the system has a unique solution. 
   \[a(-b+c) \neq 0\]

   (b) [30%] Determine $a$, $b$ and $c$ such that the system has no solution. 
   \[-b+c=0, \ a \neq 0\]

   (c) [30%] Determine $a$, $b$ and $c$ such that the system has infinitely many solutions. 
   \[a = -b+c = 0\]

   \[
   \begin{pmatrix}
   1 & b + c & a \\
   3 & 2b + 4c & a \\
   -1 & -b - c & 0 \\
   \end{pmatrix}
   \begin{pmatrix}
   a^2 - a \\
   a^2 - 2a \\
   a \\
   \end{pmatrix}
   \]

   \[
   \begin{pmatrix}
   0 & -b + c & -2a \\
   -1 & -b - c & 0 \\
   \end{pmatrix}
   \begin{pmatrix}
   a^2 \\
   -a^2 + a \\
   \end{pmatrix}
   \text{combo(1, 2, -3)}
   \]

   \[
   \begin{pmatrix}
   0 & -b + c & -2a \\
   0 & 0 & a \\
   \end{pmatrix}
   \begin{pmatrix}
   a^3 \\
   -a^2 + a \\
   \end{pmatrix}
   \text{combo(1, 3, 1)}
   \]

   \[
   \begin{pmatrix}
   0 & -b + c & 0 \\
   0 & 0 & a \\
   \end{pmatrix}
   \begin{pmatrix}
   a^2 \\
   a \\
   \end{pmatrix}
   \text{combo(3, 2, 2)}
   \]

   \[
   \begin{pmatrix}
   0 & 2b & a \\
   0 & 0 & a \\
   0 & 0 & a \\
   \end{pmatrix}
   \begin{pmatrix}
   a^2 - a \\
   a \\
   a \\
   \end{pmatrix}
   \text{when } -b+c = 0
   \]

   - Many sols in $a=0$
   No sol (signal eq $0 = a$) when $a \neq 0$
   Last eq is not a signal equation. It says $0=0$
   or else $z = a$.

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (Vector Spaces) Do all parts.

(a) [10%] True or false: There is a subspace \( S \) of \( \mathbb{R}^3 \) which does not contain the zero vector.

(b) [10%] True or false: The set of solutions \( \mathbf{x} \) in \( \mathbb{R}^3 \) of a matrix equation \( A\mathbf{x} = \mathbf{0} \) is a subspace of \( \mathbb{R}^3 \).

(c) [10%] True or false: Equations \( xy = 0 \), \( y + z = 0 \) define a subspace in \( \mathbb{R}^3 \).

(d) [10%] True or false: Equations \( x + 3y = 0 \), \( 2y + z = 0 \) define a subspace in \( \mathbb{R}^3 \).

(e) [10%] State one theorem, without proof, that concludes that a subset \( S \) of a vector space \( V \) is a subspace of \( V \).

(f) [50%] Find a basis of 4-vectors for the subspace of \( \mathbb{R}^4 \) given by the system of restriction equations

\[
\begin{align*}
2x_1 + 10x_2 - 3x_3 + 4x_4 &= 0, \\
x_1 + 4x_2 - 2x_3 + 2x_4 &= 0, \\
4x_2 + 2x_3 &= 0, \\
2x_1 + 12x_2 - 2x_3 + 4x_4 &= 0.
\end{align*}
\]

(e) If \( S = \text{span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_k \} \), then \( S \) is a subspace of \( V \).

Kernel Theorem. Thm 2 in 4.2.

Subspace criterion. Thm 1 in 4.2.

(f) \( \text{rref} \left( \begin{array}{cccc|c}
2 & 10 & -3 & 4 & 0 \\
1 & 4 & -2 & 2 & 0 \\
0 & 4 & 2 & 0 & 0 \\
2 & 12 & -2 & 4 & 0 \\
\end{array} \right) = \left( \begin{array}{cccc|c}
1 & 0 & -4 & 2 & 0 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array} \right) \)

Last frame algorithm \( \Rightarrow \) \[
\begin{align*}
\begin{cases}
x_1 &= 4t_1 - 2t_2 \\
x_2 &= -\frac{1}{2}t_1 \\
x_3 &= t_1 \\
x_4 &= t_2
\end{cases}
\end{align*}
\]

Vector basis = \[
\begin{pmatrix}
4 \\
-\frac{1}{2} \\
0 \\
1
\end{pmatrix}
\begin{pmatrix}
-2 \\
0 \\
0 \\
1
\end{pmatrix}
\]

Use this page to start your solution. Attach extra pages as needed, then staple.
3. (Independence and Dependence) Do all parts.
   (a) [10%] State a dependence test for three vectors.
   (b) [40%] Let $v_1, v_2, v_2, v_4$ denote the columns of the matrix
   \[
   A = \begin{pmatrix}
   1 & 0 & 2 & 1 \\
   2 & 0 & 5 & 1 \\
   1 & 0 & 3 & 0 \\
   -2 & 0 & -6 & 0
   \end{pmatrix}
   \]
   Display the details of an independence or dependence test for the vectors $v_1, v_2, v_2, v_4$ and report the result.
   (c) [40%] Extract from the list below a largest set of independent vectors.
   \[
   v_1 = \begin{pmatrix}
   0 \\
   0 \\
   0 \\
   0
   \end{pmatrix}, \quad v_2 = \begin{pmatrix}
   2 \\
   0 \\
   0 \\
   -2
   \end{pmatrix}, \quad v_3 = \begin{pmatrix}
   0 \\
   1 \\
   0 \\
   1
   \end{pmatrix}, \quad v_4 = \begin{pmatrix}
   0 \\
   3 \\
   0 \\
   3
   \end{pmatrix}, \quad v_5 = \begin{pmatrix}
   0 \\
   0 \\
   0 \\
   2
   \end{pmatrix}, \quad v_6 = \begin{pmatrix}
   0 \\
   3 \\
   0 \\
   5
   \end{pmatrix}
   \]
   (d) [10%] Extract from the list in (c) above a largest set of dependent vectors.
   (a) Rank Test, Determinant Test, pivot removal
   $v_1, v_2, v_3$ independent fixed vectors $\iff$ rank of augmented matrix = 3
   $v_1, v_2, v_3$ dependent fixed vectors $\iff$ rank of augmented matrix $\neq 3$
   (b) $ref\left(\begin{pmatrix}
   1 & 0 & 2 & 1 \\
   2 & 0 & 5 & 1 \\
   -2 & 0 & 3 & 0
   \end{pmatrix}\right) = \begin{pmatrix}
   1 & 0 & 0 & 3 \\
   0 & 1 & -1 & 0 \\
   0 & 0 & 0 & 0
   \end{pmatrix}$ $\implies$ dependent, because rank = 2
   or, determinant = 0 $\implies$ dependent, because 0 a column of zeros.
   (c) $ref\left(\begin{pmatrix}
   0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0
   \end{pmatrix}\right) = \begin{pmatrix}
   0 & 1 & 1 & 1 & 1 \\
   0 & 0 & 0 & 1 & v_3 \\
   0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0
   \end{pmatrix}$ pivots = 2, 4
   $\{v_2, v_4\}$ = largest independent set
   (d) Non-pivots will form a dependent set of four vectors $v_1, v_3, v_5, v_6$.
   Any set containing a dependent subset is also dependent.
   The largest dependent subset = whole set = $\{v_1, v_2, v_3, v_4, v_5, v_6\}$

Use this page to start your solution. Attach extra pages as needed, then staple.
4. (Determinants) Do all parts.
   (a) [20%] State the determinant product rule.
   (b) [20%] Assume given $3 \times 3$ matrices $A$, $B$. Suppose $E_3B = E_2E_1A$ and $E_1$, $E_2$, $E_3$ are elementary matrices representing respectively a swap, a combination, and a multiply by $-1/5$. Assume $\det(A) = 2$. Find $\det(B)$.
   (c) [20%] Determine all values of $x$ for which $B^{-1}$ fails to exist, where $B$ is the transpose of the matrix
   
   \[
   A = \begin{pmatrix}
   2 & 0 & 2 \\
   3x & 0 & 10 \\
   1 & 2x - 1 & 3x + 7
   \end{pmatrix}
   \]
   (d) [40%] Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of $A^{-1}$, given $A$ below. Other methods are not acceptable.

   
   \[
   A = \begin{pmatrix}
   1 & 1 & 1 & 1 \\
   1 & 2 & 0 & 1 \\
   -1 & 0 & -1 & 1 \\
   1 & 2 & 0 & 2
   \end{pmatrix}
   \]

   (a) If $C$, $D$ are $n \times n$ matrices, then $\det(CD) = \det(C)\det(D)$.
   (b) $|E_1| = -1$, $|E_2| = 1$, $|E_3| = -1/5$

   \[
   E_3B = E_2E_1A \Rightarrow \quad |E_3||B| = |E_2||E_1||A|
   \]

   \[
   \Rightarrow \quad -\frac{1}{5} |B| = (1)(-1)(2)
   \]

   \[
   |B| = 10
   \]

   (c) $B^{-1}$ fails to exist \iff $0 = |B| = |B^T| = |(A^T)^T| = |A|$

   \[
   |A| = (-1)(2x-1)\begin{vmatrix}
   2 & 2 \\
   3x & 10
   \end{vmatrix}
   \]

   \[
   \text{and} \quad x = \frac{1}{2} \quad \text{so} \quad x = \frac{10}{3}
   \]

   (d) Entry in row 3, col 4 of $A^{-1} = \frac{\text{ cof}(A_{4,3})}{|A|} = \frac{1}{1} \begin{pmatrix}
   1 & -1 & 0 \\
   2 & 1 & -1 \\
   -1 & 0 & -1
   \end{pmatrix} (-1)

   \[
   \text{minor}
   \]

   \[
   \text{cofactor sign}
   \]

   Use this page to start your solution. Attach extra pages as needed, then staple.
5. (Linear Differential Equations) Do all parts.
(a) [20%] Solve for the general solution of $y'' + 4y' + 20y = 0$.
(b) [40%] The characteristic equation is $r^2(r^2 + 2r + 17) = 0$. Find the general solution $y$ of the linear homogeneous constant-coefficient differential equation.
(c) [20%] A second order linear homogeneous differential equation with constant coefficients has two solutions $e^{2x} \cos x$ and $e^{2x}(2 \sin x + 3 \cos x)$. What are the roots of the characteristic equation?
(d) [20%] Circle the functions which can be a solution of a linear homogeneous differential equation with constant coefficients. For example, you would circle $\cos^2 x$ because $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x)$ is a linear combination of the two solutions $1$ and $\cos(2x)$ of the third order equation whose characteristic equation has roots $0, 2i, -2i$.

$$
\begin{align*}
&\begin{array}{cccc}
\exp(2x) & \cos^2 x & \cos(\ln |x|) & \tan x \\
0 & 100 \times & 1/e & \sinh x
\end{array} \\
&\begin{array}{cccc}
\cos 3x & \exp(-x) & \sin^2 x & \sin(x^2)
\end{array}
\end{align*}
$$

(a) $r^2 + 4r + 20 = 0$

$$
(r + 2)^2 + 16 = 0 \\
r = -2 \pm 4i$

$$y = c_1 e^{-2x} \cos(4x) + c_2 e^{-2x} \sin(4x)
$$

(b) $y =$ linear combination of the atoms listed below

- $r = 0, 0, 0$: $e^{0x}$, $xe^{0x}$, $x^2e^{0x}$
- $r = -2, -2$: $e^{-2x}$, $xe^{-2x}$
- $r = -1 \pm 4i$: $e^{-x} \cos(4x), e^{-x} \sin(4x)$

(c) $e^{2x} \cos(x)$ is a atom constructed from $2 + i$. The two roots must be $2 + i$ and $2 - i$.

(d) $\sinh(x) = \frac{1}{2} e^x - \frac{1}{2} e^{-x}$ is a linear combination of atoms $e^x, e^{-x}$

$\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)$ is a linear combination of atoms $1, \cos(2x)$.