Name KEY

2250 Midterm Exam 2, Ver 1

Wednesday, 17 March, 2010

Instructions: This in-class exam is 50 minutes. Up to 30 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(The 3 Possibilities with Symbols)**
Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & b+c & a \\ 3 & 2b+4c & a \\ -1 & -b-c & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a^2-a \\ a^2-2a \\ a \end{pmatrix}$$

(a) [40%] Determine a, b and c such that the system has a unique solution. $a(-b+c) \neq 0$

(b) [30%] Determine a, b and c such that the system has no solution. $-b+c=0,\ a\neq 0$

(c) [30%] Determine a, b and c such that the system has infinitely many solutions. $a=-b+c=0$

$$\left(\begin{array}{ccc|c} 1 & b+c & a & a^2-a \\ 3 & 2b+4c & a & a^2-2a \\ -1 & -b-c & 0 & a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & b+c & a & a^2-a \\ 0 & -b+c & -2a & -2a^2+a \\ -1 & -b-c & 0 & a \end{array}\right) \quad \text{combo}(1,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & b+c & a & a^2-a \\ 0 & -b+c & -2a & -2a^2+a \\ 0 & 0 & a & a^2 \end{array}\right) \quad \text{combo}(1,3,1)$$

$$\left(\begin{array}{ccc|c} 1 & b+c & a & a^2-a \\ 0 & -b+c & 0 & a \\ 0 & 0 & a & a^2 \end{array}\right) \quad \text{combo}(3,2,2)$$

$$\left(\begin{array}{ccc|c} 1 & 2b & a & a^2-a \\ 0 & 0 & 0 & a \\ 0 & 0 & a & a^2 \end{array}\right)$$

when $-b+c=0$
∞-many sols for $a=0$
No sol (signal eq $0=a$) when $a\neq 0$
Last eq is not a signal equation! It says $0=0$ or else $z=a$.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY

2. (Vector Spaces) Do all parts.

F (a) [10%] True or false: There is a subspace S of $\mathcal{R}^3$ which does not contain the zero vector.

T (b) [10%] True or false: The set of solutions $\mathbf{x}$ in $\mathcal{R}^3$ of a matrix equation $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathcal{R}^3$.

F (c) [10%] True or false: Equations $xy = 0$, $y + z = 0$ define a subspace in $\mathcal{R}^3$.

T (d) [10%] True or false: Equations $x + 3y = 0$, $2y + z = 0$ define a subspace in $\mathcal{R}^3$.

(e) [10%] State one theorem, without proof, that concludes that a subset S of a vector space V is a subspace of V.

(f) [50%] Find a basis of 4-vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} 2x_1 & + & 10x_2 & - & 3x_3 & + & 4x_4 & = & 0, \\ x_1 & + & 4x_2 & - & 2x_3 & + & 2x_4 & = & 0, \\ & & 4x_2 & + & 2x_3 & & & = & 0, \\ 2x_1 & + & 12x_2 & - & 2x_3 & + & 4x_4 & = & 0. \end{array}$$

(e) If $S = \text{span}(v_1, \ldots, v_k)$, Then S is a subspace of V

Kernel Theorem. Thm 2 in 4.2

Subspace criterion Thm 1 in 4.2

(f) $\text{rref}\left(\begin{array}{cccc|c} 2 & 10 & -3 & 4 & 0 \\ 1 & 4 & -2 & 2 & 0 \\ 0 & 4 & 2 & 0 & 0 \\ 2 & 12 & -2 & 4 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & -4 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Last frame algorithm $\Longrightarrow$ $\begin{cases} x_1 = 4t_1 - 2t_2 \\ x_2 = -\frac{1}{2}t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$

vector basis $= \begin{pmatrix} 4 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY

3. **(Independence and Dependence)** Do **all parts.**

(a) [10%] State a dependence test for three vectors.

(b) [40%] Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_2$, $\mathbf{v}_4$ denote the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 0 & 5 & 1 \\ 1 & 0 & 3 & 0 \\ -2 & 0 & -6 & 0 \end{pmatrix}.$$

Display the details of an independence or dependence test for the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_2$, $\mathbf{v}_4$ and report the result.

(c) [40%] Extract from the list below a largest set of independent vectors.

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \\ 9 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ -1 \\ 5 \end{pmatrix}.$$

(d) [10%] Extract from the list in (c) above a largest set of dependent vectors.

(a) Rank Test, Determinant Test, pivot Theorem

v_1, v_2, v_3 independent fixed vectors $\iff$ rank of augment matrix $= 3$

v_1, v_2, v_3 dependent fixed vectors $\iff$ rank of augmented matrix $\neq 3$

(b) $\mathrm{rref}\begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 0 & 5 & 1 \\ 1 & 0 & 3 & 0 \\ -2 & 0 & -6 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow$ dependent, because rank $= 2$

or, determinant $= 0 \Longrightarrow$ dependent, because of a column of zeros.

(c) $\mathrm{rref}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 3 & 2 & -1 \\ 0 & 2 & 1 & 9 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

pivots $= 2, 4$

$\{v_2, v_4\}$ = largest indep set

(d) Non-pivots will form a dependent set of four vectors v_1, v_3, v_5, v_6.

Any set containing a dependent subset is also dependent.

The largest dependent subset = whole set $= \{v_1, v_2, v_3, v_4, v_5, v_6\}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY

4. **(Determinants)** Do all parts.

(a) [20%] State the determinant product rule.

(b) [20%] Assume given 3×3 matrices A, B. Suppose $E_3 B = E_2 E_1 A$ and E_1, E_2, E_3 are elementary matrices representing respectively a swap, a combination, and a multiply by $-1/5$. Assume $\det(A) = 2$. Find $\det(B)$.

(c) [20%] Determine all values of x for which B^{-1} fails to exist, where B is the transpose of the matrix

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 3x & 0 & 10 \\ 1 & 2x-1 & 3x+7 \end{pmatrix}.$$

(d) [40%] Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(a) If C, D are $n \times n$ matrices, Then $|CD| = |C|\,|D|$.

(b) $|E_1| = -1$, $|E_2| = 1$, $|E_3| = -1/5$

$E_3 B = E_2 E_1 A \implies |E_3||B| = |E_2||E_1||A|$

$\implies -\frac{1}{5}|B| = (1)(-1)(2)$

$\implies |B| = 10$

(c) B^{-1} fails to exist $\iff 0 = |B| = |B^T| = |(A^T)^T| = |A|$

$|A| = (-1)(2x-1)\begin{vmatrix} 2 & 2 \\ 3x & 10 \end{vmatrix} = (-1)(2x-1)(20-6x)$

ans: $x = 1/2$ or $x = 10/3$

(d) entry in row 3, col 4 of $A^{-1} = \dfrac{\text{cof}(A,4,3)}{|A|} = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix}(-1)}{|A|}$

minor

checkerboard sign

$= \dfrac{(2)}{1}(-1) = -2$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY

5. **(Linear Differential Equations)** Do all parts.
(a) [20%] Solve for the general solution of $y'' + 4y' + 20y = 0$.
(b) [40%] The characteristic equation is $r^3(r+2)^2(r^2+2r+17) = 0$. Find the general solution y of the linear homogeneous constant-coefficient differential equation.
(c) [20%] A second order linear homogeneous differential equation with constant coefficients has two solutions $e^{2x}\cos x$ and $e^{2x}(2\sin x + 3\cos x)$. What are the roots of the characteristic equation?
(d) [20%] **Circle the functions** which can be a solution of a linear homogeneous differential equation with constant coefficients. For example, you would circle $\cos^2 x$ because $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ is a linear combination of the two solutions 1 and $\cos(2x)$ of the third order equation whose characteristic equation has roots $0,\ 2i,\ -2i$.

$e^{\pi x}$ $\quad 100 \quad e^{x^2} \quad \cos(\ln|x|) \quad \tan x$

$\cos 3x \quad 1/e \quad \sinh x \quad \sin^2 x \quad \sin(x^2)$

(a) $r^2 + 4r + 20 = 0$
$(r+2)^2 + 16 = 0$
$r = -2 \pm 4i$

atoms = $e^{-2x}\cos(4x)$, $e^{-2x}\sin(4x)$

$y = c_1 e^{-2x}\cos(4x) + c_2 e^{-2x}\sin(4x)$

(b) y = linear combination of the atoms listed below

$r = 0,0,0$: $e^{0x},\ xe^{0x},\ x^2e^{0x}$
$r = -2,-2$: $e^{-2x},\ xe^{-2x}$
$r = -1 \pm 4i$: $e^{-x}\cos(4x),\ e^{-x}\sin(4x)$

(c) $e^{2x}\cos(x)$ is a atom constructed from $2+i$. The two roots must be $2+i$ and $2-i$.

(d) $\sinh(x) = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$ is a linear combination of atoms e^{x}, e^{-x}
$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$ is a linear combination of atoms $1, \cos(2x)$

Use this page to start your solution. Attach extra pages as needed, then staple.