Name KEY

Scores
1.
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4.
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Differential Equations and Linear Algebra 2250

Midterm Exam 1 [7:30]

Thursday, 18 February 2010

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

(a) [25%] Solve $y' = \dfrac{2+x^2}{2+x}$.

(b) [25%] Solve $y' = (\csc x + 1)(\csc x - 1)$.

(c) [25%] Solve $y' = 3x^2 \ln\left|100 + x^3\right|$, $y(0) = 2$.

(d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}((1+t^2)v(t)) = 0$, $v(0) = 5$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = -1000$.

(a) $\int y'dx = \int \dfrac{2+x^2}{2+x}dx$

$y = \int \left(x - 2 + \dfrac{6}{x+2}\right)dx$

$\boxed{y = \dfrac{x^2}{2} - 2x + 6\ln|x+2| + c}$

Division algorithm

$$
\begin{array}{r}
x - 2 \\
2+x \,\big)\, 2+x^2 \\
2x+x^2 \\
\hline
2-2x \\
-4-2x \\
\hline
6 = \text{remainder}
\end{array}
$$

(b) $\int y'dx = \int (\csc^2 x - 1)dx$

$\boxed{y = -\cot x - x + c}$

(c) $\int y'dx = \int 3x^2 \ln|100+x^3|dx \leftarrow \left[\begin{array}{l} u = 100+x^3 \\ du = 3x^2dx \end{array}\right.$

$y = \int \ln|u|du$

$y = u\ln|u| - u + c$

$y = (100+x^3)\ln|100+x^3| - 100 - x^3 + c$

$2 = 100\ln 100 - 100 + 0 + c$ $\leftarrow$ set $x=0$, $y=2$

$\boxed{y = (100+x^3)\ln|100+x^3| - x^3 + 2 - 100\ln(100)}$

(d) $(1+t^2)\,v(t) = c$

$(1+0)\,5 = c$ (set $t=0, v=5$)

$\boxed{v(t) = \dfrac{5}{1+t^2}}$

$x' = \dfrac{5}{1+t^2}$

$\int x'dt = 5\int \dfrac{dt}{1+t^2}$

$x(t) = 5\arctan(t) + c$

Use this page to start your solution. Attach extra pages as needed, then staple.

$-1000 = 5 \cdot 0 + c$ (set $t=0$, $x=-1000$)

$\boxed{x(t) = 5\tan^{-1}(t) - 1000}$

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2. **(Classification of Equations)**
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{\text{X}}$) the problems that can be converted into separable form. No details expected.

☒	$y' + (x^2 + y)y = y(y + e^x)$	☒	$y' = (x - 1)(y + 1) + (1 - x)y$
☒	$y' = \cos(x + y) + \sin(x)\sin(y)$	☒	$xy' = xy^2 + 2y^2$

(b) [10%] State a calculus test that decides if $y' = f(x, y)$ is a linear equation.

(c) [20%] Find F and G such that $f(x, y) = F(x)G(y)$ for $f(x, y) = e^{x+y} - 2e^y$.

(d) [30%] Apply a test to show that $y' = x^2 + y^2$ is not separable. Supply all details.

(a) $y' = (e^x - x^2)y$

$y' = x - 1$

$y' = \cos x \cos y - \sin x \sin y + \sin x \sin y$

$y' = \cos x \cos y$

$y' = y^2 + \frac{2}{x}y^2$

$y' = \left(1 + \frac{2}{x}\right)y^2$

(b) $\frac{\partial f}{\partial y}$ independent of $y \Longrightarrow$ DE is linear

(c) $f = e^x e^y - 2e^y = (e^x - 2)e^y \Longrightarrow \boxed{F = e^x - 2 \text{ and } G = e^y}$

(d) $\frac{\partial f}{\partial x} / f = \frac{2x}{x^2 + y^2}$ depends on $y \Rightarrow$ DE is not separable

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3. (Solve a Separable Equation)
Given $(x+2)(y-2)y' = \left((x+2)e^{1-x} + 6x^2 + 4\right)(y+3)(y+4)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{(y-2)y'}{(y+3)(y+4)} = e^{1-x} + \frac{6x^2+4}{x+2}$$

Division Algorithm

$$\begin{array}{r|l} & 6x-12 \\ \hline x+2 & 6x^2+4 \\ & 6x^2+12x \\ \hline & -12x+4 \\ & -12x-24 \\ \hline & 28 \end{array}$$

$$\left(\frac{A}{y+3} + \frac{B}{y+4}\right)y' = e^{1-x} + 6x - 12 + \frac{28}{x+2}$$

Partial fractions

$$\frac{y-2}{(y+3)(y+4)} = \frac{A}{y+3} + \frac{B}{y+4}$$

$$y-2 = A(y+4) + B(y+3)$$

$y=-3$: $-5 = A + 0$

$y=-4$: $-6 = 0 - B$

$$\boxed{A=-5,\ B=6}$$

Apply quadrature

$$A\ln|y+3| + B\ln|y+4| = -e^{1-x} + 3x^2 - 12x + 28\ln|x+2| + \text{constant}$$

$$\boxed{-5\ln|y+3| + 6\ln|y+4| = -e^{1-x} + 3x^2 - 12x + 28\ln|x+2| + c}$$

ans check: Differentiate w/r/t x using $y=y(x)$:

$$-\frac{5}{y+3}y' + \frac{6}{y+4}y' = e^{1-x} + 6x - 12 + \frac{28}{x+2} + 0$$

$$\left(\frac{-5(y+4)+6(y+3)}{(y+3)(y+4)}\right)y' = e^{1-x} + \frac{6(x-2)(x+2)+28}{x+2}$$

$$\frac{y-2}{(y+3)(y+4)}y' = e^{1-x} + \frac{6x^2-24+28}{x+2}$$

$$\frac{y-2}{(y+3)(y+4)}y' = e^{1-x} + \frac{6x^2+4}{x+2}$$

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4. (Linear Equations)

(a) [50%] Solve the linear model $4x'(t) = -180 + \dfrac{24}{3t+2}x(t)$, $x(0) = 30$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\dfrac{dy}{dx} - (\sin x)y = 0$.

(c) [30%] Solve $11\dfrac{dy}{dx} + 22y = 5$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $x' = -\dfrac{180}{4} + \dfrac{6}{3t+2}x$

$x' - \dfrac{6}{3t+2}x = -45$

$\dfrac{(xW)'}{W} = -45$ where $W = e^{\int \frac{-6\,dt}{3t+2}} = e^{-\frac{6}{3}\ln|3t+2|}$

choose $W = (3t+2)^{-2}$

Quadrature step

$xW = -45\int W\,dt$

$xW = -45\int (3t+2)^{-2}dt$

$x = \dfrac{c}{W} - \dfrac{45}{3}\,\dfrac{(3t+2)^{-1}}{-1}\cdot\dfrac{1}{W}$

$30 = \dfrac{c}{(0+2)^{-2}} + \dfrac{45}{3}(0+2)^{-1}\cdot\dfrac{1}{(0+2)^{-2}} \Rightarrow c = 0$

$\boxed{x(t) = 15(3t+2)}$

(b) $y = \dfrac{c}{\text{integ. fac.}} = \boxed{\dfrac{c}{e^{\cos x}}}$

(c) $y_p = \dfrac{5}{22}$ $\quad y_h = \dfrac{c}{\text{integ. fac.}} = \dfrac{c}{e^{\int \frac{22}{11}dx}} = \dfrac{c}{e^{2x}}$

$\boxed{y = y_p + y_h = \dfrac{5}{22} + ce^{-2x}}$

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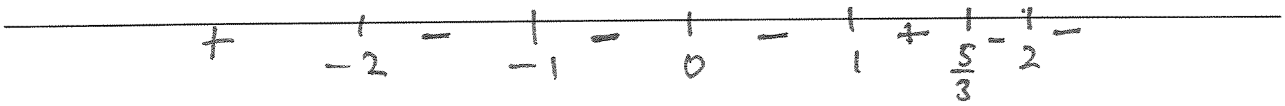
5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \ln(1+x^4)\,(4-|3x-1|)^3\,(2-x)(x^2-4)(1-x^2)^5.$$

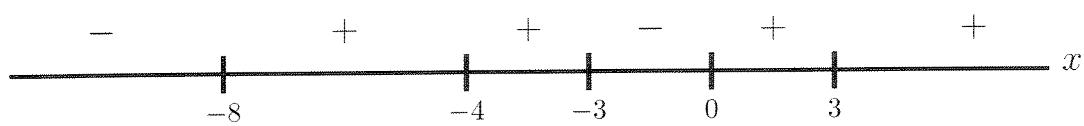
Expected in the phase line diagram are equilibrium points and signs of dx/dt.

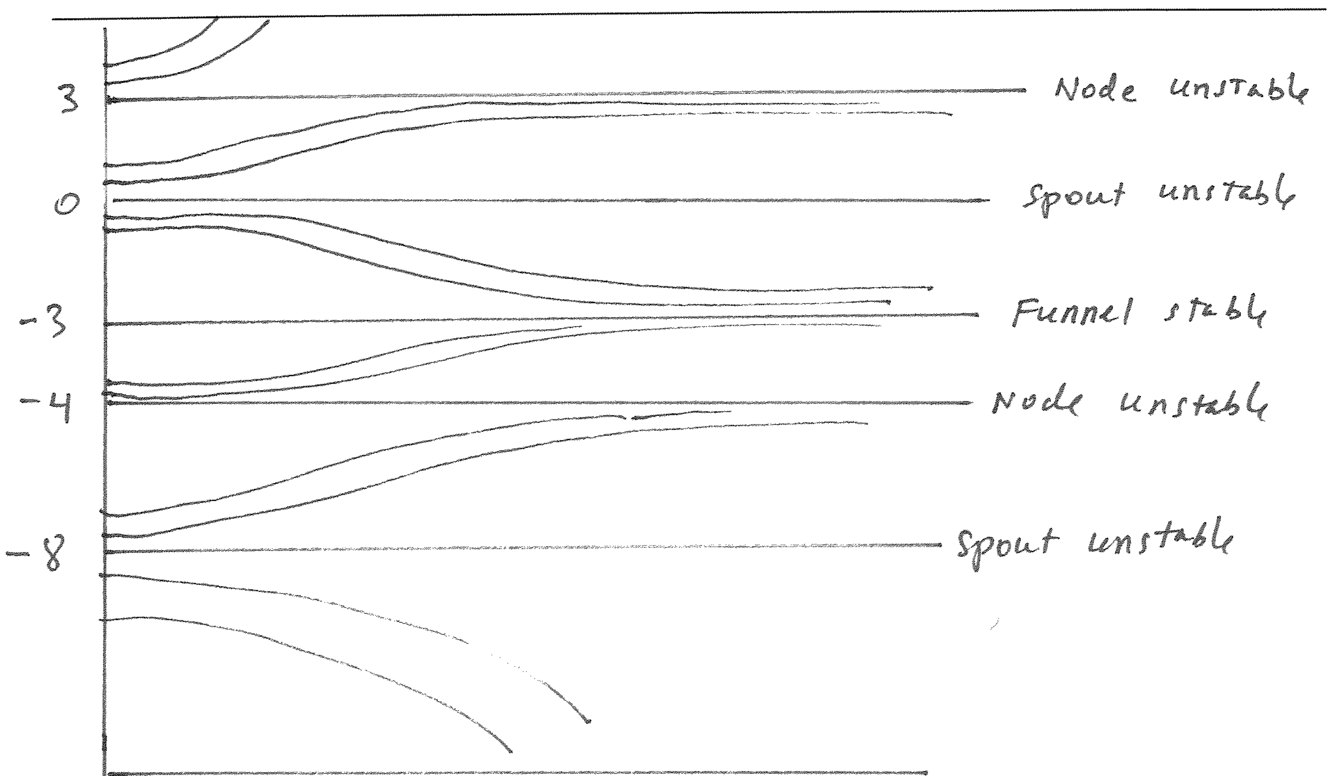


roots: $x = 0, -1, 5/3, 2, -2, 1$

signs: plus, Minus, minus, minus, plus, minus, minus

(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: **funnel**, **spout**, **node** [neither spout nor funnel], **stable**, **unstable**.





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