Name Key

Scores
1.
2.
3.
4.
5.

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [12:55]
Wednesday, 17 February 2010

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

(a) [25%] Solve $y' = \dfrac{2+x^2}{1-x}$.

(b) [25%] Solve $y' = (\sec x + 1)(\sec x - 1)$.

(c) [25%] Solve $y' = 2x \ln\left|1+x^2\right|$, $y(0) = 2$.

(d) [25%] Find the position $x(t)$ from the velocity model $\frac{d}{dt}(t^2 v(t)) = 0$, $v(1) = 20$ and the position model $\frac{dx}{dt} = v(t)$, $x(1) = -100$.

(a) $\int y' dx = \int \frac{2+x^2}{1-x} dx$

$y = \int \left(-x-1+\frac{3}{1-x}\right) dx$

$\boxed{y = -\frac{x^2}{2} - x - 3\ln|1-x| + c}$

$$\begin{array}{r} -x-1 \\ 1-x \,\overline{)\, 2+x^2} \\ -x+x^2 \\ \hline 2+x \\ -1+x \\ \hline 3 \end{array}$$

Check:

$y' = -x-1+\frac{3}{1-x}$

$= \frac{-(1+x)(1-x)+3}{1-x}$

$= \frac{-1+x^2+3}{1-x}$

$= \frac{2+x^2}{1-x}$

(b) $\int y' dx = \int (\sec^2 x - 1) dx$

$\boxed{y = \tan x - x + c}$

Check:

$y' = (\tan x - x + c)'$

$= \sec^2 x - 1$

(c) $\int y' dx = \int 2x \ln|1+x^2| dx$

$y = \int \ln|u| du \quad u = 1+x^2$

$= u\ln|u| - u + c$

$= (1+x^2)\ln(1+x^2) - 1 - x^2 + c$

$2 = (1+0^2)\ln(1+0^2) - 1 - 0^2 + c \Rightarrow c = 3$

$\boxed{y = (1+x^2)\ln(1+x^2) - x^2 + 2}$

check:

$y' = ((1+x^2)\ln(1+x^2) - x^2 + 2)'$

$= 2x\ln(1+x^2) + (1+x^2)\frac{2x}{1+x^2} - 2x + 0$

$= 2x\ln(1+x^2)$

(d) $t^2 v = c$ by quadrature, Then

$1^2 20 = c$

$v = \frac{20}{t^2}$

$x' = \frac{20}{t^2}$

$\int x' dt = \int \frac{20}{t^2} dt$

$x(t) = -\frac{20}{t} + c$

$-100 = -\frac{20}{1} + c$

$\boxed{\begin{array}{l} x(t) = -\frac{20}{t} - 80 \\ v(t) = \frac{20}{t^2} \end{array}}$

Check:

$(t^2 v)' = (20)' = 0$

$x' = \frac{20}{t^2} + 0$

$= v(t)$

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2. (Classification of Equations)
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{X}$) the problems that can be converted into separable form. No details expected.

☒ $y' + xy = y(2y + e^x) - 2y^2$	☒ $y' = (x-1)(y+1) + (1-x)y$
☒ $y' = \sin(x+y) - \sin(x)\cos(y)$	☐ $xy' = xy + y^2$

(b) [10%] State a calculus test that decides if $y' = f(x,y)$ is a quadrature equation.

(c) [20%] Apply a test to show that $y' = x^2 + y^2$ is not linear. Supply all details.

(d) [30%] Apply a test to show that $y' = x + y^2$ is not separable. Supply all details.

(a) $y' + xy = 2y^2 + e^x y - 2y^2$
$y' = (e^x - x)y$

$y' = \sin x \cos y + \cos x \sin y - \sin x \sin y$
$= \cos x \sin y$

$y' = xy - y + x - 1 + y - xy$
$= -y + x - 1 + y$
$= x - 1$

$y' = \dfrac{xy + y^2}{x} = y + \dfrac{y^2}{x}$

$\dfrac{\partial f}{\partial y}/f = \dfrac{1 + 2y/x}{y + y^2/x}$ depends on x not separable

(b) $\dfrac{\partial f}{\partial y} = 0 \Rightarrow$ Quadrature DE

(c) $\dfrac{\partial f}{\partial y}$ indep of $y \Rightarrow$ linear DE; $f(x,y) = x^2 + y^2$, $\dfrac{\partial f}{\partial y} = 2y$ is not independent of y, so the DE is not linear

(d) $\dfrac{\partial f}{\partial y}/f$ depends on x $\Rightarrow y' = f(x,y)$ is not separable.

$f(x,y) = x + y^2$

$\dfrac{\partial f}{\partial y} = 2y$

$\dfrac{\frac{\partial f}{\partial y}}{f} = \dfrac{2y}{x+y^2}$ depends on x $\Rightarrow$ DE not separable

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3. (Solve a Separable Equation)
Given $(x+2)(y-1)y' = \left((x+2)e^{1-x} + 6x^2 + 4\right)(y+1)(y+2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$(y-1)y' = \left(e^{1-x} + \frac{6x^2+4}{x+2}\right)(y+1)(y+2)$$ Divide by $x+2$

$$\frac{y-1}{(y+1)(y+2)}y' = e^{1-x} + \frac{6x^2+4}{x+2}$$ Divide by $(y+1)(y+2)$

$$\int(\text{LHS})dx = \int \frac{(y-1)y'}{(y+1)(y+2)}dx$$
$$= \int\left(\frac{A}{y+1} + \frac{B}{y+2}\right)dx$$
$$= A\ln|y+1| + B\ln|y+2| + c_1$$
$$= -2\ln|y+1| + 3\ln|y+2| + c_1$$

partial fractions

$$\frac{y-1}{(y+1)(y+2)} = \frac{A}{y+1} + \frac{B}{y+2}$$
$$y-1 = A(y+2)+B(y+1)$$
$$y=-1:\ -2 = A+0$$
$$y=-2:\ -3 = 0-B$$

$$\int(\text{RHS})dx = \int\left(e^{1-x} + \frac{6x^2+4}{x+2}\right)dx$$
$$= -e^{1-x} + \int\left(6x-12+\frac{28}{x+2}\right)dx$$
$$= -e^{1-x} + 3x^2 - 12x + 28\ln|x+2| + c_2$$

Division algorithm

$$\begin{array}{r} 6x-12 \\ x+2\ \overline{)\ 6x^2+4} \\ \underline{6x^2+12x} \\ -12x+4 \\ \underline{-12x-24} \end{array}$$

Remainder $= 28$

answer:

$$\boxed{-2\ln|y+1| + 3\ln|y+2| = -e^{1-x} + 3x^2 - 12x + 28\ln|x+2| + c}$$

Check: Differentiate, using $y=y(x)$, across the implicit solution above.

$$-2\frac{y'}{y+1} + 3\frac{y'}{y+2} = e^{1-x} + 6x - 12 + \frac{28}{x+2}$$

$$\frac{-2y-4+3y+3}{(y+1)(y+2)}y' = e^{1-x} + \frac{6x(x+2)-12(x+2)+28}{x+2}$$

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$$\frac{y-1}{(y+1)(y+2)}y' = e^{1-x} + \frac{6x^2+4}{x+2}$$

$$(x+2)(y-1)y' = \left((x+2)e^{1-x} + 6x^2+4\right)(y+1)(y+2)$$ DE verified

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4. (Linear Equations)

(a) [50%] Solve the linear model $4x'(t) = -160 + \dfrac{24}{2t+3}x(t)$, $x(0) = 30$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\dfrac{dy}{dx} - (\cos x)y = 0$.

(c) [30%] Solve $11\dfrac{dy}{dx} + 33y = 5$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $\begin{cases} x' = -40 + \dfrac{6}{2t+3}x \\ x(0) = 30 \end{cases}$

$x' + \left(\dfrac{-6}{2t+3}\right)x = -40$

$\dfrac{(xW)'}{W} = -40$

Integrating factor calculation

$W = e^{\int \frac{-6\,dt}{2t+3}}$

$= e^{-\frac{6}{2}\ln|2t+3|}$

$W = (2t+3)^{-3}$ is simplest factor

$(xW)' = -40W$

$xW = -40\int (2t+3)^{-3}dt$

$xW = (-40)\dfrac{(2t+3)^{-2}}{-2}\cdot\dfrac{1}{2} + c$

$x(0)W(0) = 20(0+3)^{-2}\cdot\dfrac{1}{2} + c$ set $t=0$, use $x(0)=30$

$30(3^{-3}) = \dfrac{10}{9} + c$

$\dfrac{10}{9} = \dfrac{10}{9} + c \implies c = 0$

answer: $x(t) = 10(2t+3)^{-2}/(2t+3)^{-3} \implies \boxed{x(t) = 10(2t+3)}$

(b) $y = \dfrac{c}{\text{integ. fac.}} = \dfrac{c}{e^{-\int \cos x\,dx}} = \boxed{\dfrac{c}{e^{-\sin x}}}$

(c) $y_p = \dfrac{5}{33}$ and $y_h = \dfrac{c}{\text{integ. fac.}} = \dfrac{c}{e^{3x}}$

$\boxed{y = \dfrac{5}{33} + ce^{-3x}}$

Check:

$y' = -3ce^{-3x}$

$= -3\left[y - \dfrac{5}{33}\right]$

$= -3y + \dfrac{5}{11}$

$11y' = -33y + 5$

$11y' + 33y = 5$

DE Verified

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Ans Check(a): $x' = (20t+30)' = 20$

$LHS = 4x' = 80$

$RHS = -160 + \dfrac{24}{2t+3}x = -160 + \dfrac{24}{2t+3}10(2t+3) = -160 + 240 = 80$ DE verified

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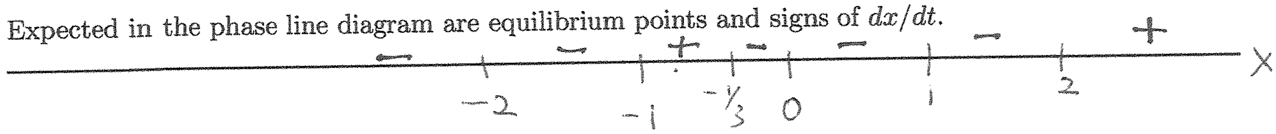
5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \ln(1+x^4)\,(2-|3x-1|)^3\,(2+x)(x^2-4)(1-x^2)^5.$$

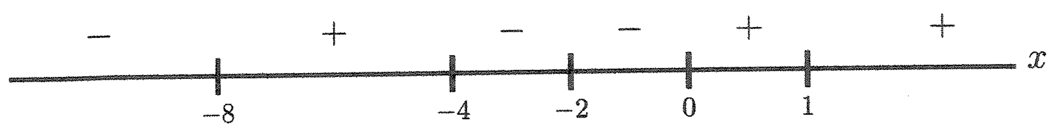
Expected in the phase line diagram are equilibrium points and signs of dx/dt.

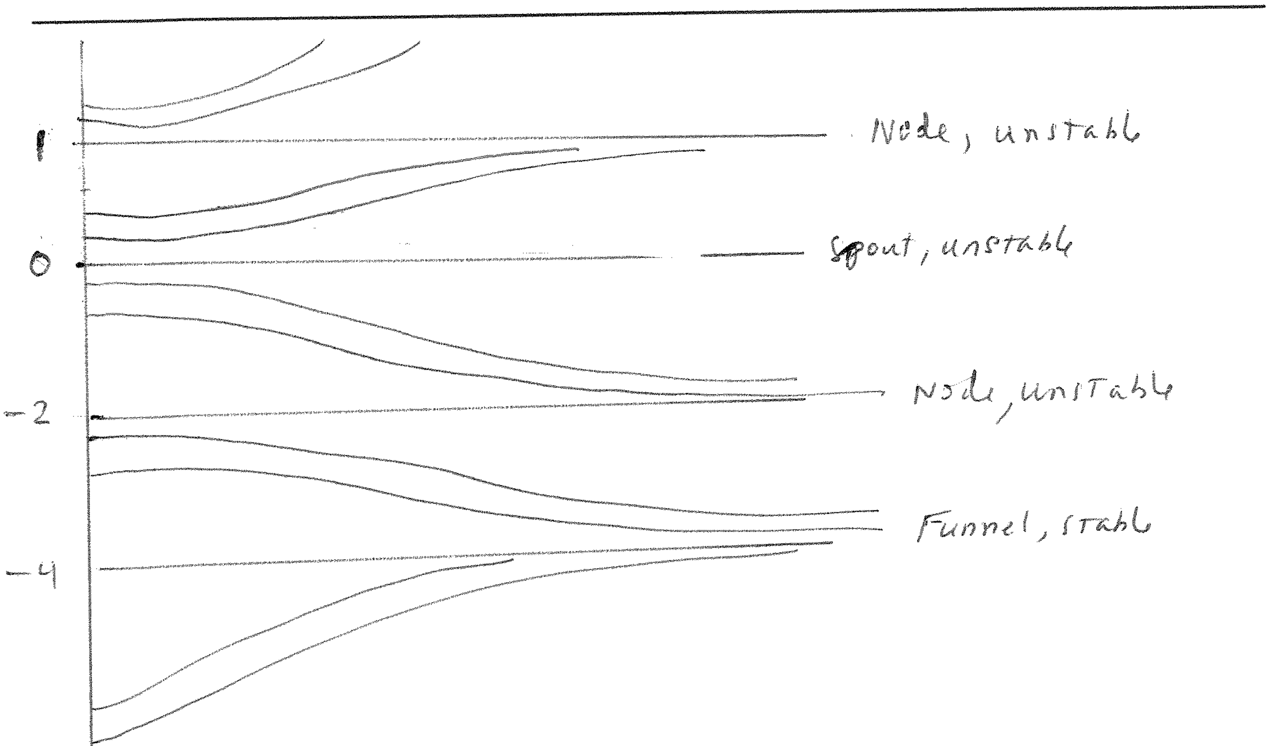


roots are $x = 0, 1, -\frac{1}{3}, -2, 2, -1$

Left-to-right signs: minus, minus, plus, minus, minus, minus, plus

(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: **funnel**, **spout**, **node** [neither spout nor funnel], **stable**, **unstable**.





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