

**Math 2250 Extra Credit Problems**  
**Chapter 8**  
**Spring 2010**

**Due date:** The due date for these problems is the day after classes end. Records are locked on that date and only corrected, never appended. Credits earned here apply only to chapter 8 and not to any other chapter.

**Submitted work.** Please submit one stapled package per problem. Kindly label problems Extra Credit. Label each problem with its corresponding problem number, e.g., Xc1.2-4. You may attach this printed sheet to simplify your work.

**Problem Xc8.1-4. (Fundamental Matrix)**

Find a fundamental matrix  $\Phi(t)$  by each of the following methods. Report  $e^{At} = \Phi(t)\Phi(0)^{-1}$ , using one of the answers for  $\Phi$ .

$$\mathbf{u}' = A\mathbf{u}, \quad A = \begin{pmatrix} 2 & -5 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

- (a) *Cayley-Hamilton method.* Compute the characteristic equation  $\det(A - \lambda I) = 0$ . Find two atoms from the roots of this equation. Then  $x(t)$  is a linear combination of these atoms. The first equation  $x' = 2x - 5y$  can be solved for  $y$  to find the second answer.
- (b) *Eigenanalysis method.* Find the eigenpairs  $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2)$  of  $A$ . Let  $\Phi$  have columns  $e^{\lambda_1 t} \mathbf{v}_1, e^{\lambda_2 t} \mathbf{v}_2$ .

**Problem Xc8.1-12. (Putzer's Method)**

The exponential matrix  $e^{At}$  can be found in the  $2 \times 2$  case from Putzer's formula

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} (A - \lambda_1 I).$$

If the roots  $\lambda_1, \lambda_2$  of  $\det(A - \lambda I) = 0$  are equal, then compute the Newton quotient factor by L'Hopital's rule, limiting  $\lambda_2 \rightarrow \lambda_1$  [ $\lambda_1, t$  fixed]. If the roots are complex, then take the real part of the right side of the equation.

Compute from Putzer's formula  $e^{At}$  for the following cases.

- (a)  $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ . Answer  $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$ .
- (b)  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .
- (c)  $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ .
- (d)  $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ .

**Problem Xc8.1-38. (Laplace's Method)**

The exponential matrix  $e^{At}$  can be found from the Laplace resolvent formula for the problem  $\Phi' = A\Phi, \Phi(0) = I$ :

$$\mathcal{L}(\Phi(t)) = (sI - A)^{-1} \Phi(0) = (sI - A)^{-1}.$$

For example,  $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$  gives  $\mathcal{L}(e^{At}) = \begin{pmatrix} s-1 & 0 \\ 0 & s-2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}(e^t) & 0 \\ 0 & \mathcal{L}(e^{2t}) \end{pmatrix}$ , which implies  $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$ .

Compute  $\Phi(t) = e^{At}$  using the resolvent formula for the following cases.

(a)  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .

(b)  $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ .

(c)  $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ .

**Problem Xc8.2-4. (Variation of Parameters)**

Use the variation of parameters formula  $\mathbf{u}_p(t) = e^{At} \int e^{-At} \mathbf{f}(t) dt$  to find a particular solution of the given system. Please use `maple` to do the indicated integration, following the example below.

(a)  $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

(b)  $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$ .

**Example:** Solve for  $\mathbf{u}_p(t)$ :  $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t),expAt(-t).f(t));
up:=simplify(expAt(t).integral);
```

**Problem Xc8.2-19. (Initial Value Problem)**

Solve the given initial value problem using a computer algebra system. Follow the example given below.

(a)  $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ,  $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

(b)  $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$ ,  $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

**Example:** Solve for  $\mathbf{u}(t)$ :  $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ ,  $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ . The answer is  $\mathbf{u} = \begin{pmatrix} -e^{-t} \\ e^{-t} - 1 \end{pmatrix}$ .

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([-1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t=0..t),expAt(-t).f(t));
up:=unapply(expAt(t).integral,t);
u0:=Vector([-1,0]);
uh:=t->expAt(t).(u0-up(0));
u:=simplify(uh(t)+up(t));
```

**End of extra credit problems chapter 8.**