Math 2250 Extra Credit Problems
Chapter 7
Spring 2010

Due date: Submit these problems by the day after classes end. Records are locked on that date and only corrected, never appended. Credits earned here apply only to chapter 7 and not to any other chapter.

Submitted work. Please submit one stapled package per problem. Kindly label problems [Extra Credit] Label each problem with its corresponding problem number, e.g., [Xc7.1-8]. You may attach this printed sheet to simplify your work.

Problem Xc7.1-8. (Transform to a first order system)
Use the position-velocity substitution $u_1 = x(t)$, $u_2 = x'(t)$, $u_3 = y(t)$, $u_4 = y'(t)$ to transform the system below into vector-matrix form $u'(t) = Au(t)$. Do not attempt to solve the system.

$$x'' - 2x' + 5y = 0, \quad y'' + 2y' - 5x = 0.$$ 

Problem Xc7.1-20a. (Dynamical systems)
Prove this result for system

$$\begin{align*}
x' &= ax + by, \\
y' &= cx + dy. 
\end{align*}$$

(a) Apply the previous problem to solve

$$\begin{align*}
x' &= 2x - y, \\
y' &= x + 2y. 
\end{align*}$$

(b) Use first order methods to solve the system

$$\begin{align*}
x' &= 2x - y, \\
y' &= 2y. 
\end{align*}$$

Problem Xc7.2-12. (General solution answer check)
(a) Verify that $x_1(t) = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $x_2(t) = e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are solutions of $x' = Ax$, where

$$A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}.$$ 

(b) Apply the Wronskian test $\det(\text{aug}(x_1, x_2)) \neq 0$ to verify that the two solutions are independent.
(c) Display the general solution of $x' = Ax$.

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Problem Xc7.2-14. (Particular solution)

(a) Find the constants \(c_1, c_2\) in the general solution

\[
x(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}
\]

satisfying the initial conditions \(x_1(0) = 4, x_2(0) = -1\).

(b) Find the matrix \(A\) in the equation \(\mathbf{x}' = A\mathbf{x}\). Use the formula \(AP = PD\) and Fourier’s model for \(A\), which is given implicitly in (a) above.

Problem Xc7.3-8. (Eigenanalysis method 2 × 2)

(a) Find \(\lambda_1, \lambda_2, v_1, v_2\) in Fourier’s model \(A(c_1 v_1 + c_2 v_2) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2\) for

\[
A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.
\]

(b) Display the general solution of \(\mathbf{x}' = A\mathbf{x}\).

Problem Xc7.3-20. (Eigenanalysis method 3 × 3)

(a) Find \(\lambda_1, \lambda_2, \lambda_3, v_1, v_2, v_3\) in Fourier’s model \(A(c_1 v_1 + c_2 v_2 + c_3 v_3) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + c_3 \lambda_3 v_3\) for

\[
A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix}.
\]

(b) Display the general solution of \(\mathbf{x}' = A\mathbf{x}\).

Problem Xc7.3-30. (Brine Tanks)

Consider two brine tanks satisfying the equations

\[
x_1'(t) = -k_1 x_1 + k_2 x_2, \quad x_2' = k_1 x_1 - k_2 x_2.
\]

Assume \(r = 10\) gallons per minute, \(k_1 = r/V_1, k_2 = r/V_2, x_1(0) = 30\) and \(x_2(0) = 0\). Let the tanks have volumes \(V_1 = 50\) and \(V_2 = 25\) gallons. Solve for \(x_1(t)\) and \(x_2(t)\).

Problem Xc7.3-40. (Eigenanalysis method 4 × 4)

Display (a) Fourier’s model and (b) the general solution of \(\mathbf{x}' = A\mathbf{x}\) for the \(4 \times 4\) matrix

\[
A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -16 & -4 \end{pmatrix}.
\]

End of extra credit problems chapter 7.