2.3 Linear Equations I

An equation $y' = f(x, y)$ is called **first-order linear** or a **linear equation** provided it can be rewritten in the special form

$$(1) \qquad y' + p(x)y = r(x)$$

for some functions $p(x)$ and $r(x)$. In most applications, p and r are assumed to be continuous. The function $p(x)$ is called the **coefficient of** y. The function $r(x)$ (r abbreviates *right side*) is called the **non-homogeneous term** or the **forcing term**. Engineering texts call $r(x)$ the **input** and the solution $y(x)$ the **output**.

A practical test:

> An equation $y' = f(x, y)$ with f continuously differentiable is **linear** provided $f_y(x, y)$ is independent of y.

Form (1) is obtained by defining $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$. Two examples:

$Ly' + Ry = E$	The LR-circuit equation with $p(x) = R/L$ and $r(x) = E/L$. Symbols L, R and E are respectively inductance, resistance and electromotive force.
$y' = -h(y - y_1)$	Newton's cooling equation with $p(x) = h$ and $r(x) = hy_1$. Oven temperature y_1 and meat thermometer reading $y(t)$ appear in the roast model.

Classifying Linear Equations

Algebraic complexity may make an equation $y' = f(x, y)$ appear to be **non-linear**, e.g., $y' = (\sin^2(xy) + \cos^2(xy))y$ simplifies to $y' = y$.

Computer algebra systems classify an equation $y' = f(x, y)$ as linear provided the identity $f(x, y) = f(x, 0) + f_y(x, 0)y$ is valid. Equivalently, $f(x, y) = r(x) - p(x)y$, where $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$. Automatic simplifications in computer algebra systems make this test practical. Hand verification can use the same method.

Elimination of an equation $y' = f(x, y)$ from the class of linear equations can be done from *necessary conditions*. The equality $f_y(x, y) = f_y(x, 0)$ implies two such conditions:

1. If $f_y(x, y)$ depends on y, then $y' = f(x, y)$ is not linear.

2. If $f_{yy}(x, y) \neq 0$, then $y' = f(x, y)$ is not linear.

For instance, either condition implies $y' = 1 + y^2$ is *not linear*.

Variation of Parameters and Integrating Factors

The initial value problem

$$y' + p(x)y = r(x), \quad y(x_0) = 0, \tag{2}$$

where p and r are continuous in an interval containing $x = x_0$, has an explicit solution (justified on page 90)

$$y(x) = \left(y_0 + \int_{x_0}^{x} r(t)e^{\int_{x_0}^{t} p(s)ds}\, dt\right) e^{-\int_{x_0}^{x} p(s)ds}. \tag{3}$$

Formula (3) is called **variation of parameters**, for historical reasons. While (3) has some appeal, applications use the **integrating factor method** below, which is developed with indefinite integrals for computational efficiency. No one memorizes (3); they remember and study the *method*. See Example 11, page 88, for technical details.

Integrating Factor Identity

The technique called the **integrating factor method** uses the replacement rule (justified on page 90)

$$\text{Fraction } \frac{(YW)'}{W} \text{ replaces } Y' + p(x)Y, \text{ where } W = e^{\int p(x)dx}. \tag{4}$$

The factor $W = e^{\int p(x)dx}$ in (4) is called an **integrating factor**.

The Integrating Factor Method

Standard Form	Rewrite $y' = f(x, y)$ in the form $y' + p(x)y = r(x)$ where p, r are continuous. The method applies only in case this is possible.
Find W	Find a simplified formula for $W = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently.
Prepare for Quadrature	Obtain the new equation $\frac{(yW)'}{W} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by equivalence (4).
Method of Quadrature	Clear fractions to obtain $(yW)' = rW$. Apply the method of quadrature to get $yW = \int r(x)W(x)dx + C$. Divide by W to isolate the explicit solution $y(x)$.

In identity (4), functions p, Y and Y' are assumed continuous with p and Y *arbitrary* functions. The integral $\int p(x)dx$ equals $P(x)+C$, where $P(x)$ is some anti-derivative of $p(x)$. Because $e^{\int p(x)dx} = e^{P(x)}e^{C}$, then factor

e^C divides out of the fraction in (4). Applications therefore simplify the **integrating factor** $e^{\int p(x)dx}$ to $e^{P(x)}$, where $P(x)$ is *any suitable antiderivative* of $p(x)$ (effectively, we take $C = 0$).

Equation (4) is central to the method, because it collapses the two terms $y' + py$ into a single term $(Wy)'/W$; the method of quadrature applies to $(Wy)' = rW$. The literature calls the exponential factor W an **integrating factor** and equivalence (4) a **factorization** of $Y' + p(x)Y$.

Simplifying an integrating factor. Factor W is simplified by dropping constants of integration. To illustrate, if $p(x) = 1/x$, then $\int p(x)dx = \ln|x| + C$. The algebra rule $e^{A+B} = e^A e^B$ implies that $W = e^C e^{\ln|x|} = |x|e^C = (\pm e^C)x$. Let $c_1 = \pm e^C$. Then $W = c_1 W_1$ where $W_1 = x$. The fraction $(Wy)'/W$ reduces to $(W_1 y)'/W_1$, because c_1 cancels. In an application, we choose the simpler expression W_1. The illustration also shows that the exponential in W can sometimes be eliminated.

Superposition

Formula (3) can be decomposed into two expressions, called y_h and y_p, so that the **general solution** is expressed as $y = y_h + y_p$. The function y_h solves the homogeneous equation $y' + p(x)y = 0$ and y_p solves the non-homogeneous equation $y' + p(x)y = r(x)$. This observation is called the **superposition principle**.

Equation (3) implies the **homogeneous solution** y_h and a **particular solution** y_p^* can be defined by

$$(5) \quad y_h = y_0 e^{-\int_{x_0}^x p(s)ds}, \quad y_p^* = \left(\int_{x_0}^x r(t)e^{\int_{x_0}^t p(s)ds}dt\right) e^{-\int_{x_0}^x p(s)ds}.$$

Verification amounts to setting $r = 0$ in (3) to determine y_h. The solution y_p^* depends on the forcing term $r(x)$, but y_h does not. Initial conditions of a problem are buried in y_h. Experimentalists view the computation of y_p^* as a *single experiment* in which the state y_p^* is determined by the forcing term $r(x)$ and zero initial data $y = 0$ at $x = x_0$.

Structure of Solutions

Formula (3), proved on page 90, directly establishes existence for the solution to the linear initial value problem (2). The proof also determines what other particular solutions might be used in the formula for a general solution:

Theorem 3 (Solution Structure)
Assume $p(x)$ and $r(x)$ are continuous on $a < x < b$ and $a < x_0 < b$. Let y_h and y_p^* be defined by equation (5). Let y be a solution of $y' + p(x)y = r(x)$ on $a < x < b$. Then y can be decomposed as $y = y_h + y_p^*$, where $y_0 = y(x_0)$.

In short, a linear equation has the solution structure *homogeneous plus particular*. Two solutions of the non-homogeneous equation therefore differ by some solution y_h of the homogeneous equation.

Examples

11 Example (Integrating Factor Method) Solve $2y' + 6y = e^{-x}$.

Solution: The solution is $y = \frac{1}{4}e^{-x} + ce^{-3x}$. An answer check appears in Example 13. The details:

$y' + 3y = 0.5e^{-x}$	Divide by 2 to get the standard form.
$W = e^{3x}$	Find the integrating factor $W = e^{\int 3dx}$.
$\dfrac{\left(e^{3x}y\right)'}{e^{3x}} = 0.5e^{-x}$	Replace the LHS of $y' + 3y = 0.5e^{-x}$ by the integrating factor quotient; see page 86.
$\left(e^{3x}y\right)' = 0.5e^{2x}$	Clear fractions. Prepared for quadrature
$e^{3x}y = 0.5\int e^{2x}dx$	Method of quadrature applied.
$y = 0.5\left(e^{2x}/2 + c_1\right)e^{-3x}$	Evaluate the integral. Divide by $W = e^{3x}$.
$\quad = \frac{1}{4}e^{-x} + ce^{-3x}$	Final answer, $c = 0.5c_1$.

12 Example (Superposition) Find a particular solution of $y' + 2y = 3e^x$ with fewest terms.

Solution: The answer is $y = e^x$. The first step solves the equation using the integrating factor method, giving $y = e^x + ce^{-2x}$; details below. A particular solution with fewest terms, $y = e^x$, is found by setting $c = 0$. The solution y_p^* of equation (5) has two terms: $y_p^* = e^x - e^{3x_0}e^{-2x}$. The reason for the extra term is the condition $y = 0$ at $x = 0$. The two particular solutions differ by the homogeneous solution y_0e^{-2x} where $y_0 = e^{3x_0}$.

Integrating factor method details:

$y' + 2y = 3e^x$	The standard form.
$W = e^{2x}$	Find the integrating factor $W = e^{\int 2dx}$.
$\dfrac{\left(e^{2x}y\right)'}{e^{3x}} = 3e^x$	Integrating factor identity applied to $y' + 2y = 3e^x$.
$e^{2x}y = 3\int e^{3x}dx$	Clear fractions and apply quadrature.
$y = \left(e^{3x} + c\right)e^{-2x}$	Evaluate the integral. Isolate y.
$\quad = e^x + ce^{-2x}$	Solution found.

13 Example (Answer Check) Show the answer check details for $2y' + 6y = e^{-x}$ and candidate solution $y = \frac{1}{4}e^{-x} + ce^{-3x}$.

Solution: **Details**:

LHS $= 2y' + 6y$	Left side of the equation $2y' + 6y = e^{-x}$.
$= 2(-\frac{1}{4}e^{-x} - 3ce^{-3x}) + 6(\frac{1}{4}e^{-x} + ce^{-3x})$	Substitute for y.
$= e^{-x} + 0$	Simplify terms.
$=$ RHS	DE verified.

14 Example (Finding y_h and y_p) Find the homogeneous solution y_h and a particular solution y_p for the equation $2xy' + y = 4x^2$ on $x > 0$.

Solution: The solution by the integrating factor method is $y = 0.8x^2 + cx^{-1/2}$; details below. Then $y_h = cx^{-1/2}$ and $y_p = 0.8x^2$ give $y = y_h + y_p$.

The symbol y_p stands for *any* particular solution. It should be free of any arbitrary constants c.

Variation of parameters gives a *different* particular solution $y_p^* = 0.8x^2 - 0.8x_0^{5/2}x^{-1/2}$. It differs from the other particular solution $0.8x^2$ by a homogeneous solution $Kx^{-1/2}$.

Integrating factor method details:

$y' + 0.5y/x = 2x$	Standard form. Divided by $2x$.
$W = e^{0.5\int dx/x}$	The integrating factor is $W = e^{\int p}$.
$= e^{0.5 \ln x}$	Simplify the integration constant.
$= x^{1/2}$	Used $\ln u^n = n \ln u$. Simplified W found.
$\frac{\left(x^{1/2}y\right)'}{x^{1/2}} = 2x$	Integrating factor identity applied on the left.
$x^{1/2}y = 2\int x^{3/2}dx$	Clear fractions. Apply quadrature.
$y = \left(4x^{5/2}/5 + c\right)x^{-1/2}$	Evaluate the integral. Divide to isolate y.
$= 4x^2 + cx^{-1/2}$	Solution found.

15 Example (Classification) Classify the equation $y' = x + \ln(xe^y)$ as linear or non-linear.

Solution: It's linear, with standard linear form $y' + (-1)y = x + \ln x$. To explain why, the term $\ln(xe^y)$ on the right expands into $\ln x + \ln e^y$, which in turn is $\ln x + y$, using logarithm rules. Because $e^y > 0$, then $\ln(xe^y)$ makes sense for only $x > 0$. Henceforth, assume $x > 0$.

Computer algebra test $f(x, y) = f(x, 0) + f_y(x, 0)y$. Expected is LHS $-$ RHS $= 0$ after simplification. This example produced $\ln e^y - y$ instead of 0, evidence that limitations may exist.

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assume(x>0):
f:=(x,y)->x+ln(x*exp(y)):
LHS:=f(x,y):
RHS:=f(x,0)+subs(y=0,diff(f(x,y),y))*y:
simplify(LHS-RHS);
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If the test *passes*, then $y' = f(x, y)$ becomes $y' = f(x, 0) + f_y(x, 0)y$. This example gives $y' = x + \ln x + y$, which converts to the standard linear form $y' + (-1)y = x + \ln x$.

Details and Proofs

Justification of Formula (3): Define

$$\mathcal{Q}(x) = e^{-\int_{x_0}^{x} p(s)ds}, \quad \mathcal{R}(x) = \int_{x_0}^{x} \frac{r(t)}{\mathcal{Q}(t)}dt.$$

The calculus rule $(e^u)' = u'e^u$ and the fundamental theorem of calculus result $(\int_{x_0}^{x} G(t)dt)' = G(x)$ can be used to obtain the formulas

$$\mathcal{Q}' = (-p)\mathcal{Q}, \quad \mathcal{R}' = \frac{r}{\mathcal{Q}}.$$

Existence. Equation (3) is $y = \mathcal{Q}(y_0 + \mathcal{R})$. Existence will be established by showing that y satisfies $y' + py = r$, $y(x_0) = y_0$. The initial condition $y(x_0) = y_0$ follows from $\mathcal{Q}(x_0) = 1$ and $\mathcal{R}(x_0) = 0$. The steps below verify $y' + py = r$, completing existence.

$y' = [\mathcal{Q}(y_0 + \mathcal{R})]'$	Equation (3), using notation $\mathcal{Q}$ and $\mathcal{R}$.
$= \mathcal{Q}'(y_0 + \mathcal{R}) + \mathcal{Q}\mathcal{R}'$	Sum and product rules applied.
$= -p\mathcal{Q}(y_0 + \mathcal{R}) + \mathcal{Q}\mathcal{R}'$	Used $\mathcal{Q}' = (-p)\mathcal{Q}$.
$= -p\mathcal{Q}(y_0 + \mathcal{R}) + r$	Used $\mathcal{R}' = r/\mathcal{Q}$.
$= -py + r$	Apply $y = \mathcal{Q}(y_0 + \mathcal{R})$.

Uniqueness. It remains to show that the solution given by (3) is the only solution. Start by assuming Y is another, subtract them to obtain $u = y - Y$. Then $u' + pu = 0$, $u(x_0) = 0$. To show $y \equiv Y$, it suffices to show $u \equiv 0$.

According to the integrating factor method, the equation $u' + pu = 0$ is equivalent to $(uW)' = 0$ where $W = e^{\mathbf{P}}$ and $\mathbf{P}(x) = \int_{x_0}^{x} p(t)dt$. Integrate $(uW)' = 0$ from x_0 to x, giving $u(x)W(x) = u(x_0)W(x_0)$. Since $u(x_0) = 0$ and $W(x) \neq 0$, it follows that $u(x) = 0$ for all x. This completes the proof.

Remarks on Picard's Theorem. The Picard-Lindelöf theorem, page 62, implies existence-uniqueness, but only on a smaller interval, and furthermore it supplies no practical formula for the solution. Formula (3) is therefore an improvement over the results obtainable from the general theory.

Justification of Factorization (4): It is assumed that $Y(x)$ is a given but otherwise arbitrary differentiable function. Equation (4) will be justified in its fraction-free form

(6) $$\left(Ye^{\mathbf{P}}\right)' = (Y' + pY)\,e^{\mathbf{P}}, \quad \mathbf{P}(x) = \int p(x)dx.$$

$\text{LHS} = \left(Ye^{\mathbf{P}}\right)'$	The left side of equation (6).

$= Y'e^{\mathbf{P}} + \left(e^{\mathbf{P}}\right)' Y$	Apply the product rule $(uv)' = u'v + uv'$.
$= Y'e^{\mathbf{P}} + pe^{\mathbf{P}}Y$	Use the chain rule $(e^u)' = u'e^u$ and $\mathbf{P}' = p$.
$= (Y' + pY)\, e^{\mathbf{P}}$	The common factor is $e^{\mathbf{P}}$.
$=$ RHS	The right hand side of equation (6).

Exercises 2.3

Integrating Factor Method. Apply the integrating factor method, page 86, to solve the given linear equation. See the examples starting on page 88 for details.

1. $y' + y = e^{-x}$

2. $y' + y = e^{-2x}$

3. $2y' + y = e^{-x}$

4. $2y' + y = e^{-2x}$

5. $2y' + y = 1$

6. $3y' + 2y = 2$

7. $2xy' + y = x$

8. $3xy' + y = 3x$

9. $y' + 2y = e^{2x}$

10. $2y' + y = 2e^{x/2}$

11. $y' + 2y = e^{-2x}$

12. $y' + 4y = e^{-4x}$

13. $2y' + y = e^{-x}$

14. $2y' + y = e^{-2x}$

15. $4y' + y = 1$

16. $4y' + 2y = 3$

17. $2xy' + y = 2x$

18. $3xy' + y = 4x$

19. $y' + 2y = e^{-x}$

20. $2y' + y = 2e^{-x}$

Superposition. Find a particular solution with fewest terms. See Example 12, page 88.

21. $3y' = x$

22. $3y' = 2x$

23. $y' + y = 1$

24. $y' + 2y = 2$

25. $2y' + y = 1$

26. $3y' + 2y = 1$

27. $y' - y = e^x$

28. $y' - y = xe^x$

29. $xy' + y = \sin x$ $(x > 0)$

30. $xy' + y = \cos x$ $(x > 0)$

31. $y' + y = x - x^2$

32. $y' + y = x + x^2$

General Solution. Find y_h and a particular solution y_p. Report the general solution $y = y_h + y_p$. See Example 14, page 89.

33. $y' + y = 1$

34. $xy' + y = 2$

35. $y' + y = x$

36. $xy' + y = 2x$

37. $y' - y = x + 1$

38. $xy' - y = 2x - 1$

39. $2xy' + y = 2x^2$ $(x > 0)$

40. $xy' + y = 2x^2$ $(x > 0)$

Classification. Classify as linear or non-linear. Use the test $f(x,y) = f(x,0) + f_y(x,0)y$ and a computer algebra system, when available, to check the answer. See Example 15, page 89.

41. $y' = 1 + 2y^2$

42. $y' = 1 + 2y^3$

43. $yy' = (1+x)\ln e^y$

44. $yy' = (1+x)(\ln e^y)^2$

45. $y'\sec^2 y = 1 + \tan^2 y$

46. $y' = \cos^2(xy) + \sin^2(xy)$

47. $y'(1+y) = xy$

48. $y' = y(1+y)$

49. $xy' = (x+1)y - xe^{\ln y}$

50. $2xy' = (2x+1)y - xye^{-\ln y}$

Proofs and Details.

51. Prove directly without appeal to Theorem 3 that the difference of two solutions of $y' + p(x)y = r(x)$ is a solution of the homogeneous equation $y' + p(x)y = 0$.

52. Prove that y_p^* given by equation (5) and $y_p = Q^{-1}\int r(x)Q(x)dx$ given in the integrating factor method are related by $y_p = y_p^* + y_h$ for some solution y_h of the homogeneous equation.

53. Let $Q' = 1 + x$ and define $y = Q^{-1}\int e^x Q(x)dx$. Show directly, without appeal to theorems of this section, that $y' + (1+x)y = e^x$.

54. Let $Q' = x\ln(1+x^2)$ and define $y = Q^{-1}\int x^2 Q(x)dx$. Show directly, without appeal to theorems of this section, that $y' + x\ln(1+x^2)y = x^2$.