2.5 Linear Applications

This collection of applications for the linear equation $y' + p(x)y = r(x)$ includes mixing problems, especially brine tanks in single and multiple cascade, heating and cooling problems based upon Newton's law of cooling, radioactive isotope chains, and elementary electric circuits.

Developed here is the theory for mixing cascades, heating and cooling. Radioactive decay theory was developed on page 3. Electric circuits of type LR or RC were developed on page 16.

Brine Mixing

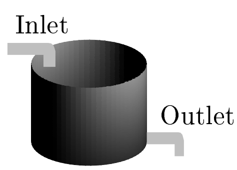


Figure 1. A brine tank.
The tank has one inlet and one outlet. The inlet supplies a brine mixture and the outlet drains the tank.

A given tank contains brine, that is, water and salt. Input pipes supply other, possibly different brine mixtures at varying rates, while output pipes drain the tank. The problem is to determine the salt $x(t)$ in the tank at any time.

The basic chemical law to be applied is the **mixture law**

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}.$$

The law is applied under a simplifying assumption: *the concentration of salt in the brine is uniform throughout the fluid.* Stirring is one way to meet this requirement. Because of the uniformity assumption, the amount $x(t)$ of salt in kilograms divided by the volume $V(t)$ of the tank in liters gives salt **concentration**[2] $x(t)/V(t)$ kilograms per liter.

One Input and One Output. Let the input be $a(t)$ liters per minute with concentration C_1 kilograms of salt per liter. Let the output empty $b(t)$ liters per minute. The tank is assumed to contain V_0 liters of brine at $t = 0$. The tank gains fluid at rate $a(t)$ and loses fluid at rate $b(t)$, therefore $V(t) = V_0 + \int_0^t [a(r) - b(r)]dr$ is the volume of brine in the tank at time t. The *mixture law* applies to obtain (derived on page 107) the model linear differential equation

$$\frac{dx}{dt} = C_1 a(t) - \frac{b(t)x(t)}{V(t)}. \tag{1}$$

[2]Concentration is defined as amount per unit volume.

Two-Tank Mixing. Two tanks A and B are assumed to contain A_0 and B_0 liters of brine at $t = 0$. Let the input for the first tank A be $a(t)$ liters per minute with concentration C_1 kilograms of salt per liter. Let tank A empty at $b(t)$ liters per minute into a second tank B, which itself empties at $c(t)$ liters per minute.

Let $x(t)$ be the number of kilograms of salt in tank A at time t. Similarly, $y(t)$ is the amount of salt in tank B. The *objective* is to find differential equations for the unknowns $x(t)$, $y(t)$.

Fluid loses and gains in each tank give rise to the brine volume formulas $V_A(t) = A_0 + \int_0^t [a(r) - b(r)]dr$ and $V_B(t) = B_0 + \int_0^t [b(r) - c(r)]dr$, respectively, for tanks A and B, at time t.

The *mixture law* applies to obtain the model linear differential equations

$$\begin{aligned} \frac{dx}{dt} &= C_1 a(t) - \frac{b(t)x(t)}{V_A(t)}, \\ \frac{dy}{dt} &= \frac{b(t)x(t)}{V_A(t)} - \frac{c(t)y(t)}{V_B(t)}. \end{aligned}$$

The first equation was solved in the previous paragraph, hence there is an explicit formula for $x(t)$. Substitute this formula into the second equation, then solve for $y(t)$ (by the same method).

Residential Heating and Cooling

The internal temperature $u(t)$ in a residence fluctuates with the outdoor temperature, indoor heating and indoor cooling. Newton's law of cooling can be written in this case as

$$\text{(2)} \qquad \frac{du}{dt} = k(a(t) - u(t)) + s(t) + f(t),$$

where the various symbols have the interpretation below.

k	The insulation constant: $k \approx 1/4$ for good insulation and $k \approx 1/2$ for no insulation.
$a(t)$	The ambient outside temperature.
$s(t)$	Combined rate for all inside heat sources. Includes living beings, appliances and whatever uses energy.
$f(t)$	Inside heating or cooling rate.

A derivation of (2) appears on page 107. To solve equation (2), write it in standard linear form and use the integrating factor method on page 84.

No Sources. Assume the absence of heating inside the building, that is, $s(t) = f(t) = 0$. Let the outside temperature be constant: $a(t) = a_0$. Equation (2) simplifies to the Newton cooling equation on page 4:

$$\frac{du}{dt} + ku(t) = ka_0. \tag{3}$$

From Theorem 1, page 4, the solution is

$$u(t) = a_0 + (u(0) - a_0)e^{-kt}. \tag{4}$$

This formula represents *exponential decay* of the interior temperature from $u(0)$ to a_0.

Half-Time Insulation Constant. Suppose it's 50°F outside and 70°F initially inside, when the electricity goes off. How long does it take to drop to 60°F inside? The answer is *about 1–3 hours*, depending on the insulation.

The importance of 60°F is that it is halfway between the inside and outside temperatures of 70°F and 50°F. The range 1–3 hours is found from (4) by solving $u(T) = 60$ for T, in the extreme cases of poor or excellent insulation.

The more general equation $u(T) = (a_0 + u(0))/2$ can be solved. The answer is $T = \ln(2)/k$, called the **half-time insulation constant** for the residence. It measures the insulation quality, larger T corresponding to better insulation. For most residences, the half-time insulation constant ranges from 1.4 to 2.8 hours.

Winter Heating. The introduction of a furnace and a thermostat set at temperature T_0 (typically, 68°F to 72°F) changes the source term $f(t)$ to the special form

$$f(t) = k_1(T_0 - u(t)),$$

according to Newton's law of cooling, where k_1 is a constant. The differential equation (2) becomes

$$\frac{du}{dt} = k(a(t) - u(t)) + s(t) + k_1(T_0 - u(t)). \tag{5}$$

It is a first-order linear differential equation which can be solved by the integrating factor method.

Summer Air Conditioning. An air conditioner used with a thermostat leads to the same differential equation (5) and solution, because Newton's law of cooling applies to both heating and cooling.

Evaporative Cooling. In desert-mountain areas, where summer humidity is low, the **evaporative cooler** is a popular low-cost solution to cooling. The cooling effect is due to heat loss from the supply of outside air, caused by energy conversion during water evaporation. Cool air is pumped into the residence much like a furnace pumps warm air. An evaporative cooler may have no thermostat. The temperature $P(t)$ of the pumped air depends on the outside air temperature and humidity.

A Newton's cooling model for the inside temperature $u(t)$ requires a constant k_1 for the evaporative cooling term $f(t) = k_1(P(t) - u(t))$. If $s(t) = 0$ is assumed, then equation (2) becomes

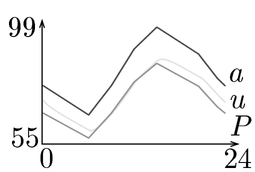
$$\frac{du}{dt} = k(a(t) - u(t)) + k_1(P(t) - u(t)). \tag{6}$$

This is a first-order linear differential equation, solvable by the integrating factor method.

During hot summer days the relation $P(t) = 0.85a(t)$ could be valid, that is, the air pumped from the cooler vent is 85% of the ambient outside temperature $a(t)$. Extreme temperature variations can occur in the fall and spring. In July, the reverse is possible, e.g., $100 < a(t) < 115$. Assuming $P(t) = 0.85a(t)$, the solution of (6) is

$$u(t) = u(0)e^{-kt-k_1t} + (k + 0.85k_1)\int_0^t a(r)e^{(k+k_1)(r-t)}dr.$$

Figure 2 shows the solution for a 24-hour period, using a sample profile $a(t)$, $k = 1/4$, $k_1 = 2$ and $u(0) = 69$. The residence temperature $u(t)$ is expected to be approximately between $P(t)$ and $a(t)$.



$$a(t) = \begin{cases} 75 - 2t & 0 \le t \le 6 \\ 39 + 4t & 6 < t \le 9 \\ 30 + 5t & 9 < t \le 12 \\ 54 + 3t & 12 < t \le 15 \\ 129 - 2t & 15 < t \le 21 \\ 170 - 4t & 21 < t \le 23 \\ 147 - 3t & 23 < t \le 24 \end{cases}$$

Figure 2. A 24-hour plot of P, u and temperature profile $a(t)$.

Examples

18 Example (Pollution) When industrial pollution in Lake Erie ceased, the level was five times that of its inflow from Lake Huron. Assume Lake Erie has perfect mixing, constant volume V and equal inflow/outflow rates of $0.73V$ per year. Estimate the time required to reduce the pollution in half.

Solution: The answer is about 1.34 years. An overview of the solution will be given, followed by technical details.

Overview. The brine-mixing model applies to pollution problems, giving a differential equation model for the pollution concentration $x(t)$,

$$x'(t) = 0.73Vc - 0.73x(t), \quad x(0) = 5cV,$$

where c is the inflow pollution concentration. The model has solution

$$x(t) = x(0)\left(0.2 + 0.8e^{-0.73t}\right).$$

Solving for the time T at which $x(T) = \frac{1}{2}x(0)$ gives $T = \ln(8/3)/0.73 = 1.34$ years.

Model details. The rate of change of $x(t)$ equals the concentration rate in minus the concentration rate out. The in-rate equals c times the inflow rate, or $c(0.73V)$. The out-rate equals $x(t)$ times the outflow rate, or $\frac{0.73V}{V}x(t)$. This justifies the differential equation. The statement $x(0)$=“five times that of Lake Huron” means that $x(0)$ equals $5c$ times the volume of Lake Erie, or $5cV$.

Solution details. Re-write the differential equation as $x'(t) + 0.73x(t) = 0.73x(0)/5$. It has equilibrium solution $x_p = x(0)/5$. The homogeneous solution is $x_h = ke^{-0.73t}$, from the theory of growth-decay equations. Adding x_h and x_p gives the general solution x. To solve the initial value problem, substitute $t = 0$ and find $k = 4x(0)/5$. Substitute for k into $x = x(0)/5 + ke^{-0.73t}$ to obtain the reported solution.

Equation for T details. The equation $x(T) = \frac{1}{2}x(0)$ becomes $x(0)(0.2 + 0.8e^{-0.73T}) = x(0)/2$, which by algebra reduces to the exponential equation $e^{-0.73T} = 3/8$. Take logarithms to isolate $T = -\ln(3/8)/0.73 \approx 1.3436017$.

19 Example (Brine Cascade) Assume brine tanks A and B in Figure 3 have volumes 100 and 200 gallons, respectively. Let $A(t)$ and $B(t)$ denote the number of pounds of salt at time t, respectively, in tanks A and B. Pure water flows into tank A, brine flows out of tank A and into tank B, then brine flows out of tank B. All flows are at 4 gallons per minute. Given $A(0) = 40$ and $B(0) = 40$, find $A(t)$ and $B(t)$.

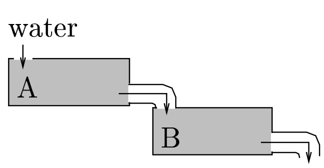


Figure 3. Cascade of two brine tanks.

Solution: The solutions for the brine cascade are (details below)

$$A(t) = 40e^{-t/25}, \quad B(t) = 120e^{-t/50} - 80e^{-t/25}.$$

Modeling. This is an instance of the two-tank mixing problem on page 99. The volumes in the tanks do not change and the input salt concentration is $C_1 = 0$. The equations are

$$\frac{dA}{dt} = -\frac{4A(t)}{100}, \quad \frac{dB}{dt} = \frac{4A(t)}{100} - \frac{4B(t)}{200}.$$

Solution $A(t)$ details.

$A' = -0.04A, \quad A(0) = 40$	Initial value problem to be solved.
$A = 40e^{-t/25}$	Solution found by the growth-decay recipe.

Solution $B(t)$ details.

$B' = 0.04A - 0.02B, \quad B(0) = 40$	Initial value problem to be solved.
$B' + 0.02B = 1.6e^{-t/25}$	Substitute for A. Get standard form.
$B' + 0.02B = 0, \quad B(0) = 40$	Homogeneous problem to be solved.
$B_h = 40e^{-t/50}$	Homogeneous solution. Growth-decay recipe applied.
$B_p = e^{-t/50}\int_0^t 1.6e^{-r/25}e^{r/50}dr$	Variation of parameters solution.
$\quad = 80e^{-t/50} - 80e^{-t/25}$	Evaluate integral.
$B = B_h + B_p$	Superposition.
$\quad = 120e^{-t/50} - 80e^{-t/25}$	Final solution.

The solution can be checked in `maple` as follows.

```
de1:=diff(x(t),t)=-4*x(t)/100:
de2:=diff(y(t),t)=4*x(t)/100-4*y(t)/200:
ic:=x(0)=40,y(0)=40:
dsolve({de1,de2,ic},{x(t),y(t)});
```

20 Example (Office Heating) A worker shuts off the office heat and goes home at 5PM. It's 72°F inside and 60°F outside overnight. Estimate the office temperature at 8PM, 11PM and 6AM.

Solution:

The temperature estimates are 62.7-65.7°F, 60.6-62.7°F and 60.02-60.5°F. Details follow.

Model. The residential heating model applies, with no sources, to give $u(t) = a_0 + (u(0) - a_0)e^{-kt}$. Supplied are values $a_0 = 60$ and $u(0) = 72$. Unknown is constant k in the formula

$$u(t) = 60 + 12e^{-kt}.$$

Estimation of k. To make the estimate for k, assume the range $1/4 \le k \le 1/2$, which covers the possibilities of poor to excellent insulation.

Calculations. The estimates requested are for $t = 3$, $t = 6$ and $t = 13$. The formula $u(t) = 60 + 12e^{-kt}$ and the range $0.25 \le k \le 0.5$ gives the estimates

$$\begin{aligned} 62.68 \le 60 + 12e^{-3k} &\le 65.67, \\ 60.60 \le 60 + 12e^{-6k} &\le 62.68, \\ 60.02 \le 60 + 12e^{-13k} &\le 60.47. \end{aligned}$$

21 Example (Spring Temperatures) It's spring. The outside temperatures are between 45°F and 75°F and the residence has no heating or cooling. Find an approximation for the interior temperature fluctuation $u(t)$ using the estimate $a(t) = 60 - 15\cos(\pi(t-4)/12)$, $k = \ln(2)/2$ and $u(0) = 53$.

Solution: The approximation, justified below, is

$$u(t) \approx -8.5e^{-kt} + 60 + 1.5\cos\frac{\pi t}{12} - 12\sin\frac{\pi t}{12}.$$

Model. The residential model for no sources applies. Then

$$u'(t) = k(a(t) - u(t)).$$

Computation of $u(t)$. Let $\omega = \pi/12$ and $k = \ln(2)/2$. The solution is

$u = u(0)e^{-kt} + \int_0^t ka(r)e^{k(r-t)}dr$	Variation of parameters.
$= 53e^{-kt} + \int_0^t 15k(4 - \cos\omega(t-4))e^{k(r-t)}dr$	Insert $a(t)$ and $u(0)$.
$\approx -8.5e^{-kt} + 60 + 1.5\cos\omega t - 12\sin\omega t$	Used `maple` integration.

The `maple` code used for the integration appears below.

```
k:=ln(2)/2: u0:=53:
F:=r->k*(60-15*cos(Pi *(r-4)/12)):
A:=t->(u0+int(F(r)*exp(k*r),r=0..t))*exp(-k*t);
simplify(A(t));
```

22 Example (Temperature Variation) Justify that in the spring and fall, the interior of a residence has temperature variation between 69% and 89% of the outside temperature variation.

Solution: The justification necessarily makes some assumptions, which are:

$a(t) = B - A\cos\omega(t-4)$	Assume $A > 0$, $B > 0$, $\omega = \pi/12$ and extreme temperatures at 4AM and 4PM.
$s(t) = 0$	No inside heat sources.
$f(t) = 0$	No furnace or air conditioner.
$1/4 \le k \le 1/2$	Vary from excellent to poor insulation.
$u(0) = B$	The average of the outside low and high.

Model. The residential model for no sources applies. Then

$$u'(t) = k(a(t) - u(t)).$$

Formula for u. Variation of parameters gives a compact formula:

$u = u(0)e^{-kt} + \int_0^t ka(r)e^{k(r-t)}dr$	See (5), page 85.
$= Be^{-kt} + \int_0^t k(B - A\cos\omega(t-4))e^{k(r-t)}dr$	Insert $a(t)$ and $u(0)$.
$= c_0Ae^{-kt} + B + c_1A\cos\omega t + c_2A\sin\omega t$	Evaluate. Values below.

The values of the constants in the calculation of u are

$$c_0 = 72k^2 - 6k\pi\sqrt{3}, \quad c_1 = \frac{6k\pi\sqrt{3} - 72k^2}{144k^2 + \pi^2}, \quad c_2 = \frac{-6k\pi - 72k^2\sqrt{3}}{144k^2 + \pi^2}.$$

The trigonometric formula $a\cos\theta + b\sin\theta = r\sin(\theta + \phi)$ where $r^2 = a^2 + b^2$ and $\tan\phi = a/b$ can be applied to the formula for u to rewrite it as

$$u = c_0 A e^{-kt} + B + A\sqrt{c_1^2 + c_2^2}\,\sin(\omega t + \phi).$$

The outside low and high are $B - A$ and $B + A$. The outside temperature variation is their difference $2A$. The exponential term contributes less than one degree after 12 hours. The inside low and high are therefore approximately $B - rA$ and $B + rA$ where $r = \sqrt{c_1^2 + c_2^2}$. The inside temperature variation is their difference $2rA$, which is r times the outside variation.

It remains to show that $0.69 \le r \le 0.89$. The equation for r has a simple representation:

$$r = \frac{12k}{\sqrt{144k^2 + \pi^2}}.$$

It has derivative $dr/dk > 0$. The extrema occur at the endpoints of the interval $1/4 \le k \le 1/2$, giving values 0.69 and 0.89, approximately. This justifies the estimates of 69% and 89%.

The `maple` code used for the integration appears below.

```
omega:=Pi/12:
F:=r->k*(B-A*cos(omega *(r-4))):
G:=t->(B+int(F(r)*exp(k*r),r=0..t))*exp(-k*t);
simplify(G(t));
```

23 Example (Radioactive Chain) Let A, B and C be the amounts of three radioactive isotopes. Assume A decays into B at rate a, then B decays into C at rate b. Given $a \ne b$, $A(0) = A_0$ and $B(0) = 0$, find formulas for A and B.

Solution: The isotope amounts are (details below)

$$A(t) = A_0 e^{-at}, \quad B(t) = aA_0 \frac{e^{-at} - e^{-bt}}{b - a}.$$

Modeling. The reaction model will be shown to be

$$A' = -aA,\ A(0) = A_0, \quad B' = aA - bB,\ B(0) = 0.$$

The derivation uses the radioactive decay law on page 18. The model for A is simple decay $A' = -aA$. Isotope B is *created* from A at a rate equal to the disintegration rate of A, or aA. But B itself undergoes disintegration at rate bB. The rate of increase of B is not aA but the difference of aA and bB, which accounts for lost material. Therefore, $B' = aA - bB$.

Solution Details for A.

$A' = -aA, \quad A(0) = A_0$	Initial value problem to solve.
$A = A_0 e^{-at}$	Use the *growth-decay recipe* on page 3.

Solution Details for B.

$B' = aA - bB, \quad B(0) = 0$	Initial value problem to solve.
$B' + bB = aA_0 e^{-at}, \quad B(0) = 0$	Insert $A = A_0 e^{-at}$. Standard form.
$B = e^{-bt} \int_0^t aA_0 e^{-ar} e^{br} dr$	Variation of parameters solution y_p^*, page 91. It already satisfies $B(0) = 0$.
$= aA_0 \dfrac{e^{-at} - e^{-bt}}{b-a}$	Evaluate the integral for $b \neq a$.

24 Example (Electric Circuits) For the LR-circuit of Figure 4, show that $I_{SS} = E/R$ and $I_{tr} = I_0 e^{-Rt/L}$ are the steady-state and transient currents.

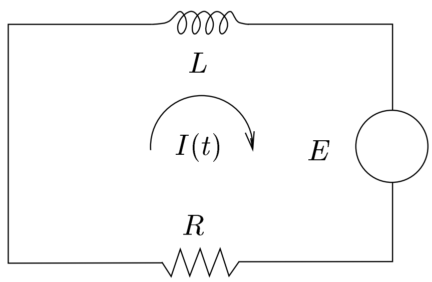


Figure 4. An LR-circuit with constant voltage E and zero initial current $I(0) = 0$.

Solution:

Model. The LR-circuit equation is derived from Kirchhoff's laws and the voltage drop formulas on page 17. The only new element is the added electromotive force term $E(t)$, which is set equal to the algebraic sum of the voltage drops, giving the model

$$LI'(t) + RI(t) = E(t), \quad I(0) = I_0.$$

General solution. The details:

$I' + (R/L)I = E/L$	Standard linear form.
$I_p = E/R$	Set I=constant, solve for a particular solution I_p.
$I' + (R/L)I = 0$	Homogeneous equation. Solve for $I = I_h$.
$I_h = I_0 e^{-Rt/L}$	Growth-decay recipe, page 4.
$I = I_h + I_p$	Superposition.
$= I_0 e^{-Rt/L} + E/R$	General solution found.

Steady-state solution. The steady-state solution is found by striking out from the general solution all terms that approach zero at $t = \infty$. Remaining after strike-out is $I_{SS} = E/R$.

Transient solution. The term *transient* refers to the terms in the general solution which approaches zero at $t = \infty$. Therefore, $I_{tr} = I_0 e^{-Rt/L}$.

25 Example (Time constant) Show that the current $I(t)$ in the LR-circuit of Figure 4 is at least 95% of the steady-state current E/R after three time constants, i.e., after time $t = 3L/R$.

Solution: Physically, the **time constant** L/R for the circuit is found by an experiment in which the circuit is initialized to $I = 0$ at $t = 0$, then the current I is observed until it reaches 63% of its steady-state value.

Time to 95% of I_{SS}. The solution is $I(t) = E(1 - e^{-Rt/L})/R$. Solving the inequality $1 - e^{-Rt/L} \geq 0.95$ gives

$0.95 \leq 1 - e^{-Rt/L}$	Inequality to be solved for t.
$e^{-Rt/L} \leq 1/20$	Move terms across the inequality.
$\ln e^{-Rt/L} \leq \ln(1/20)$	Take the logarithm across the inequality.
$-Rt/L \leq \ln 1 - \ln 20$	Apply logarithm rules.
$t \geq L\ln(20)/R$	Isolate t on one side.

The value $\ln(20) = 2.9957323$ leads to the rule: *after three times the time constant has elapsed, the current has reached 95% of the steady-state current.*

Details and Proofs

Brine-Mixing One-tank Proof: The brine-mixing equation $x'(t) = C_1 a(t) - b(t)x(t)/V(t)$ is justified for the one-tank model, by applying the *mixture law* "$dx/dt =$ input rate $-$ output rate" as follows.

$$\begin{aligned}\text{input rate} &= \left(a(t)\frac{\text{liters}}{\text{minute}}\right)\left(C_1\frac{\text{kilograms}}{\text{liter}}\right)\\ &= C_1 a(t)\frac{\text{kilograms}}{\text{minute}},\\ \text{output rate} &= \left(b(t)\frac{\text{liters}}{\text{minute}}\right)\left(\frac{x(t)}{V(t)}\frac{\text{kilograms}}{\text{liter}}\right)\\ &= \frac{b(t)x(t)}{V(t)}\frac{\text{kilograms}}{\text{minute}}.\end{aligned}$$

Residential Heating and Cooling Proof: Newton's law of cooling will be applied to justify the residential heating and cooling equation

$$\frac{du}{dt} = k(a(t) - u(t)) + s(t) + f(t).$$

Let $u(t)$ be the indoor temperature. The heat flux is due to three heat source rates:

$N(t) = k(a(t) - u(t))$	The Newton cooling rate.
$s(t)$	Combined rate for all inside heat sources.
$f(t)$	Inside heating or cooling rate.

The expected change in u is the sum of the rates N, s and f. In the limit, $u'(t)$ is on the left and the sum $N(t) + s(t) + f(t)$ is on the right. This completes the proof.

Exercises 2.5

Concentration. A lab assistant collects n liters of brine, boils it until only salt crystals remain, then uses a scale to determine the crystal mass m kilograms.
(a) Report the **concentration** units.
(b) Find the brine **concentration**.

1. $n = 1$, $m = 0.2275$

2. $n = 1.75$, $m = 0.32665$

3. $n = 1.5$, $m = 0.0155$

4. $n = 1.25$, $m = 0.0104$

5. $n = 2$, $m = 0.1$

6. $n = 2.5$, $m = 0.2215$

One-Tank Mixing. Assume one inlet and one outlet. Determine the amount $x(t)$ of salt in the tank at time t. Use the text notation for equation (1).

7. The inlet adds 10 liters per minute with concentration $C_1 = 0.023$ kilograms per liter. The tank contains 110 liters of distilled water. The outlet drains 10 liters per minute.

8. The inlet adds 12 liters per minute with concentration $C_1 = 0.0205$ kilograms per liter. The tank contains 200 liters of distilled water. The outlet drains 12 liters per minute.

9. The inlet adds 10 liters per minute with concentration $C_1 = 0.0375$ kilograms per liter. The tank contains 200 liters of brine in which 3 kilograms of salt is dissolved. The outlet drains 10 liters per minute.

10. The inlet adds 12 liters per minute with concentration $C_1 = 0.0375$ kilograms per liter. The tank contains 500 liters of brine in which 7 kilograms of salt is dissolved. The outlet drains 12 liters per minute.

11. The inlet adds 10 liters per minute with concentration $C_1 = 0.1075$ kilograms per liter. The tank contains 1000 liters of brine in which k kilograms of salt is dissolved. The outlet drains 10 liters per minute.

12. The inlet adds 14 liters per minute with concentration $C_1 = 0.1124$ kilograms per liter. The tank contains 2000 liters of brine in which k kilograms of salt is dissolved. The outlet drains 14 liters per minute.

13. The inlet adds 10 liters per minute with concentration $C_1 = 0.104$ kilograms per liter. The tank contains 100 liters of brine in which 0.25 kilograms of salt is dissolved. The outlet drains 11 liters per minute. Determine additionally the time when the tank is empty.

14. The inlet adds 16 liters per minute with concentration $C_1 = 0.01114$ kilograms per liter. The tank contains 1000 liters of brine in which 4 kilograms of salt is dissolved. The outlet drains 20 liters per minute. Determine additionally the time when the tank is empty.

15. The inlet adds 10 liters per minute with concentration $C_1 = 0.1$ kilograms per liter. The tank contains 500 liters of brine in which k kilograms of salt is dissolved. The outlet drains 12 liters per minute. Determine additionally the time when the tank is empty.

16. The inlet adds 11 liters per minute with concentration $C_1 = 0.0156$ kilograms per liter. The tank contains 700 liters of brine in which k kilograms of salt is dissolved. The outlet drains 12 liters per minute. Determine additionally the time when the tank is empty.

Two-Tank Mixing. Assume brine tanks A and B in Figure 3 have volumes 100 and 200 gallons, respectively. Let $A(t)$ and $B(t)$ denote the number of pounds of salt at time t, respectively, in tanks A and B. Distilled water flows into tank A, then brine flows out of tank A and into tank B, then out of tank B. All flows are at r gallons per minute. Given rate r and initial salt amounts $A(0)$ and $B(0)$, find $A(t)$ and $B(t)$.

17. $r = 4$, $A(0) = 40$, $B(0) = 20$.

18. $r = 3$, $A(0) = 10$, $B(0) = 15$.

19. $r = 5$, $A(0) = 20$, $B(0) = 40$.

20. $r = 5$, $A(0) = 40$, $B(0) = 30$.

21. $r = 8$, $A(0) = 10$, $B(0) = 12$.

22. $r = 8$, $A(0) = 30$, $B(0) = 12$.

23. $r = 9$, $A(0) = 16$, $B(0) = 14$.

24. $r = 9$, $A(0) = 22$, $B(0) = 10$.

25. $r = 7$, $A(0) = 6$, $B(0) = 5$.

26. $r = 7$, $A(0) = 13$, $B(0) = 26$

Residential Heating. Assume the Newton cooling model for heating and insulation values $1/4 \le k \le 1/2$. Follow Example 20, page 103.

27. The office heat goes off at 7PM. It's 74°F inside and 58°F outside overnight. Estimate the office temperature at 10PM, 1AM and 6AM.

28. The office heat goes off at 6:30PM. It's 73°F inside and 55°F outside overnight. Estimate the office temperature at 9PM, 3AM and 7AM.

29. The radiator goes off at 9PM. It's 74°F inside and 58°F outside overnight. Estimate the room temperature at 11PM, 2AM and 6AM.

30. The radiator goes off at 10PM. It's 72°F inside and 55°F outside overnight. Estimate the room temperature at 2AM, 5AM and 7AM.

31. The office heat goes on in the morning at 6:30AM. It's 57°F inside and 40° to 55°F outside until 11AM. Estimate the office temperature at 8AM, 9AM and 10AM. Assume the furnace provides a five degree temperature rise in 30 minutes and the thermostat is set for 76°F.

32. The office heat goes on at 6AM. It's 55°F inside and 43° to 53°F outside until 10AM. Estimate the office temperature at 7AM, 8AM and 9AM. Assume the furnace provides a seven degree temperature rise in 45 minutes and the thermostat is set for 78°F.

33. The hot water heating goes on at 6AM. It's 55°F inside and 50° to 60°F outside until 10AM. Estimate the room temperature at 7:30AM. Assume the radiator provides a four degree temperature rise in 45 minutes and the thermostat is set for 74°F.

34. The hot water heating goes on at 5:30AM. It's 54°F inside and 48° to 58°F outside until 9AM. Estimate the room temperature at 7AM. Assume the radiator provides a five degree temperature rise in 45 minutes and the thermostat is set for 74°F.

35. A portable heater goes on at 7AM. It's 45°F inside and 40° to 46°F outside until 11AM. Estimate the room temperature at 9AM. Assume the heater provides a two degree temperature rise in 30 minutes and the thermostat is set for 90°F.

36. A portable heater goes on at 8AM. It's 40°F inside and 40° to 45°F outside until 11AM. Estimate the room temperature at 10AM. Assume the heater provides a two degree temperature rise in 20 minutes and the thermostat is set for 90°F.

Evaporative Cooling. Define outside temperature (see Figure 2)

$$a(t) = \begin{cases} 75 - 2t & 0 \le t \le 6 \\ 39 + 4t & 6 < t \le 9 \\ 30 + 5t & 9 < t \le 12 \\ 54 + 3t & 12 < t \le 15. \\ 129 - 2t & 15 < t \le 21 \\ 170 - 4t & 21 < t \le 23 \\ 147 - 3t & 23 < t \le 24 \end{cases}$$

Given k, k_1, $P(t) = wa(t)$ and $u(0) = 69$, then plot $u(t)$, $P(t)$ and $a(t)$ on one graphic.

$$u(t) = u(0)e^{-kt-k_1 t} + (k + wk_1)\int_0^t a(r)e^{(k+k_1)(r-t)}dr.$$

37. $k = 1/4$, $k_1 = 2$, $w = 0.85$

38. $k = 1/4$, $k_1 = 1.8$, $w = 0.85$

39. $k = 3/8$, $k_1 = 2$, $w = 0.85$

40. $k = 3/8$, $k_1 = 2.4$, $w = 0.85$

41. $k = 1/4$, $k_1 = 3$, $w = 0.80$

42. $k = 1/4$, $k_1 = 4$, $w = 0.80$

43. $k = 1/2$, $k_1 = 4$, $w = 0.80$

44. $k = 1/2$, $k_1 = 5$, $w = 0.80$

45. $k = 3/8$, $k_1 = 3$, $w = 0.80$

46. $k = 3/8$, $k_1 = 4$, $w = 0.80$

Radioactive Chain. Let A, B and C be the amounts of three radioactive isotopes. Assume A decays into B at rate a, then B decays into C at rate b. Given a, b, $A(0) = A_0$ and $B(0) = B_0$, find formulas for A and B.

47. $a = 2$, $b = 3$, $A_0 = 100$, $B_0 = 10$

48. $a = 2$, $b = 3$, $A_0 = 100$, $B_0 = 100$

49. $a = 1$, $b = 4$, $A_0 = 100$, $B_0 = 200$

50. $a = 1$, $b = 4$, $A_0 = 300$, $B_0 = 100$

51. $a = 4$, $b = 3$, $A_0 = 100$, $B_0 = 100$

52. $a = 4$, $b = 3$, $A_0 = 100$, $B_0 = 200$

53. $a = 6$, $b = 1$, $A_0 = 600$, $B_0 = 100$

54. $a = 6$, $b = 1$, $A_0 = 500$, $B_0 = 400$

55. $a = 3$, $b = 1$, $A_0 = 100$, $B_0 = 200$

56. $a = 3$, $b = 1$, $A_0 = 400$, $B_0 = 700$

Electric Circuits. In the LR-circuit of Figure 4, assume $E(t) = A\cos wt$ and $I(0) = 0$. Solve for $I(t)$.

57. $A = 100$, $w = 2\pi$, $R = 1$, $L = 2$

58. $A = 100$, $w = 4\pi$, $R = 1$, $L = 2$

59. $A = 100$, $w = 2\pi$, $R = 10$, $L = 1$

60. $A = 100$, $w = 2\pi$, $R = 10$, $L = 2$

61. $A = 5$, $w = 10$, $R = 2$, $L = 3$

62. $A = 5$, $w = 4$, $R = 3$, $L = 2$

63. $A = 15$, $w = 2$, $R = 1$, $L = 4$

64. $A = 20$, $w = 2$, $R = 1$, $L = 3$

65. $A = 25$, $w = 100$, $R = 5$, $L = 15$

66. $A = 25$, $w = 50$, $R = 5$, $L = 5$