Systems of Differential Equations and Laplace's Method

- Solving $\boldsymbol{x}' = C\boldsymbol{x}$
- The Resolvent
- An Illustration for $\mathbf{x}' = C\mathbf{x}$

Solving $x' = Cx$

Apply $\boldsymbol{L}$ to each side to obtain $\boldsymbol{L}(\mathrm{x}') = \boldsymbol{C}\boldsymbol{L}(\mathrm{x})$. Use the parts rule

$$\boldsymbol{L}(\mathrm{x}') = \boldsymbol{s}\boldsymbol{L}(\boldsymbol{x}) - \mathrm{x}(0)$$

to obtain

$$\begin{aligned}
\boldsymbol{s}\boldsymbol{L}(\mathrm{x}) - \mathrm{x}(0) &= \boldsymbol{L}(\boldsymbol{C}\mathrm{x}) \\
\boldsymbol{s}\boldsymbol{L}(\mathrm{x}) - \boldsymbol{L}(\boldsymbol{C}\mathrm{x}) &= \mathrm{x}(0) \\
\boldsymbol{s}\boldsymbol{I}\,\boldsymbol{L}(\mathrm{x}) - \boldsymbol{C}\,\boldsymbol{L}(\mathrm{x}) &= \mathrm{x}(0) \\
(\boldsymbol{s}\boldsymbol{I} - \boldsymbol{C})\boldsymbol{L}(\mathrm{x}) &= \mathrm{x}(0).
\end{aligned}$$

Resolvent

The inverse of $\boldsymbol{s}\boldsymbol{I} - \boldsymbol{C}$ is called the **resolvent**, a term invented to describe the equation

$$\boldsymbol{L}(\mathrm{x}(t)) = (\boldsymbol{s}\boldsymbol{I} - \boldsymbol{C})^{-1}\mathrm{x}(0).$$

An Illustration for $\mathbf{x}' = C\mathbf{x}$

Define $C = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which gives a scalar initial value problem

$$\begin{cases} x_1'(t) &= 2x_1(t) &+ 3x_2(t), \\ x_2'(t) &= & 4x_2(t), \\ x_1(0) &= 1, & \\ x_2(0) &= 2. & \end{cases}$$

Then the adjugate formula $A^{-1} = \dfrac{\mathrm{adj}(A)}{\det(A)}$ gives the resolvent

$$(sI - C)^{-1} = \frac{1}{(s-2)(s-4)} \begin{pmatrix} s-4 & 3 \\ 0 & s-2 \end{pmatrix}.$$

The Laplace transform of the solution is then

$$\boldsymbol{L}(\mathbf{x}(\boldsymbol{t})) = (\boldsymbol{sI} - \boldsymbol{C})^{-1}\begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{s}+\mathbf{2}}{(\boldsymbol{s}-\mathbf{2})(\boldsymbol{s}-\mathbf{4})} \\ \dfrac{\mathbf{2}}{\boldsymbol{s}-\mathbf{4}} \end{pmatrix}.$$

Partial fractions and use of the backward Laplace table imply

$$\mathbf{x}(\boldsymbol{t}) = \begin{pmatrix} \mathbf{3}\boldsymbol{e}^{\mathbf{4}\boldsymbol{t}} - \mathbf{2}\boldsymbol{e}^{\mathbf{2}\boldsymbol{t}} \\ \mathbf{2}\boldsymbol{e}^{\mathbf{4}\boldsymbol{t}} \end{pmatrix}.$$