Phase Portraits for $\vec{u}' = A\vec{u}$

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How to Construct a Phase Portrait

Equilibria   Plot in the xy-plane all equilibria of $\vec{u}' = \vec{f}(\vec{u})$.

Window   Select an x-range and a y-range for the graph window which includes all significant equilibria (Figure 3).

Grid   Plot a uniform grid of N grid points ($N \approx 50$ for hand work) within the graph window, to populate the graphical white space (Figure 4). The isocline method might also be used to select grid points.

Field   Draw at each grid point a short tangent vector, a **replacement curve** for a solution curve through a grid point on a small time interval (Figure 5).

Orbits   Draw additional threaded trajectories on long time intervals into the remaining white space of the graphic (Figure 6). This is guesswork, based upon tangents to threaded trajectories matching nearby field tangents drawn in the previous step. See Figure 1 for matching details.
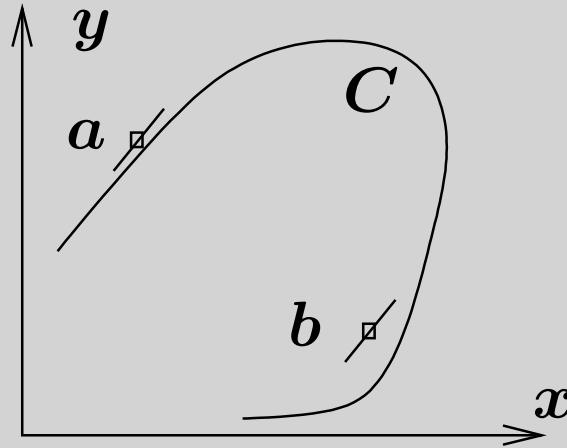
Badly Threaded Solution Curves ──────────────────────────────────

Figure 1. Badly threaded orbit.

Threaded solution curve C correctly matches its tangent to the tangent at nearby grid point a, but it fails to match at grid point b.

Solution Curve Tangent Matching

Why does a threaded solution curve tangent $\vec{T}_1$ have to match a tangent $\vec{T}_2$ at a nearby grid point (see Figure 2)? A tangent vector is given by $\vec{T} = \vec{u}'(t) = \vec{f}(\vec{u}(t))$. Then $\vec{T}_1 = \vec{f}(\vec{u}_1), \vec{T}_2 = \vec{f}(\vec{u}_2)$. However, $\vec{u}_1 \approx \vec{u}_2$ in the graphic, hence by continuity of $\vec{f}$ it follows that $\vec{T}_1 \approx \vec{T}_2$.
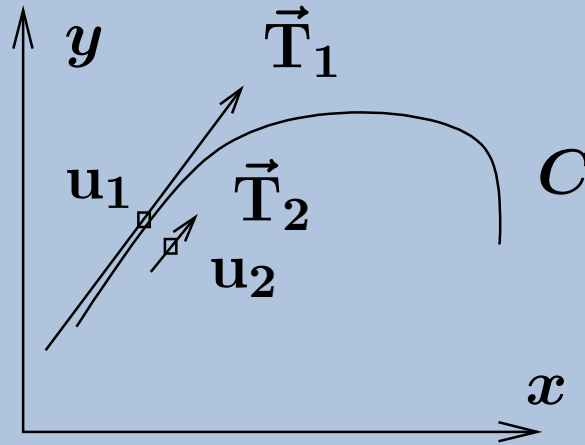

Figure 2. Tangent matching.

Threaded solution curve C matches its tangent $\vec{T}_1$ at u_1 to direction field tangent $\vec{T}_2$ at nearby grid point u_2.

Phase Portrait Illustration

Consider the illustration

$$(1) \quad \begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

- The equilibria are $(1, -1)$ and $(-1, 1)$.
- The graph window is selected as $|x| \leq 2$, $|y| \leq 2$, in order to include both equilibria.
- The uniform grid will be 11×11, although for hand work 5×5 is normal.
- Tangents at the grid points are short line segments which do not touch each another – they are graphically the same as short solution curves.
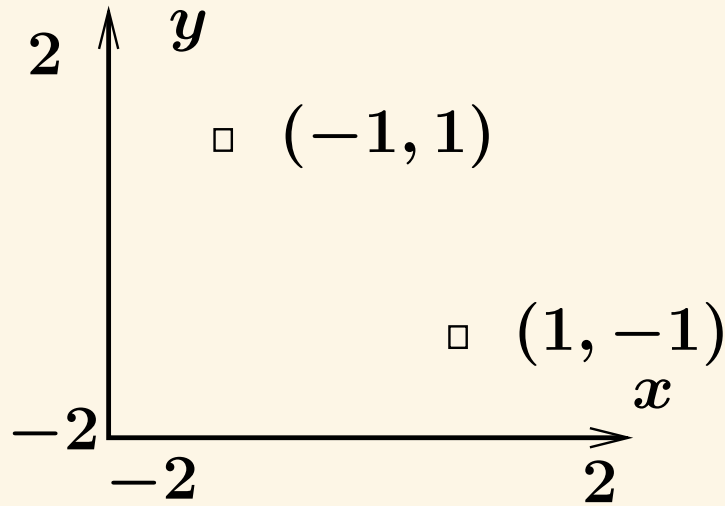
Figure 3. Equilibria $(1, -1)$, $(-1, 1)$ for (1) and graph window.

The equilibria (x, y) are calculated from equations $0 = x + y$, $0 = 1 - x^2$. The graph window $|x| \leq 2$, $|y| \leq 2$ is invented initially, then updated until Figure 5 reveals sufficiently rich field details.
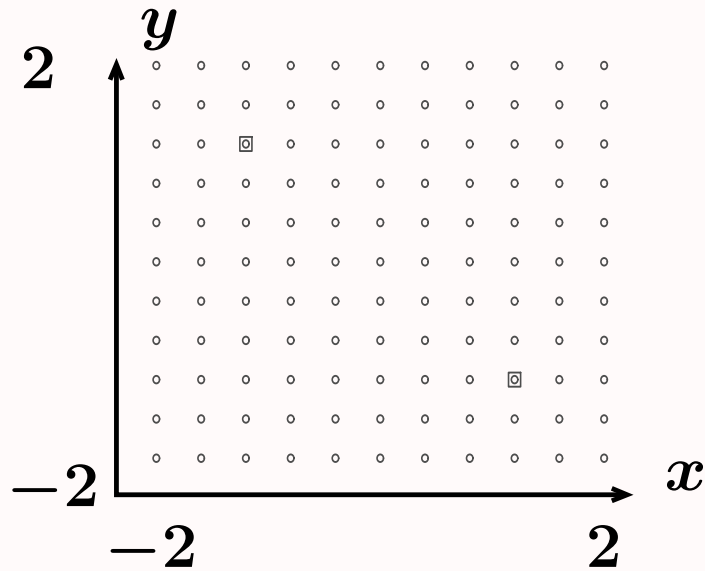
Figure 4. Equilibria $(1, -1)$, $(-1, 1)$ for (1) with 11×11 uniform grid.

The equilibria (squares) happen to cover up two grid points (circles). The size 11×11 is invented to fill the white space in the graphic.
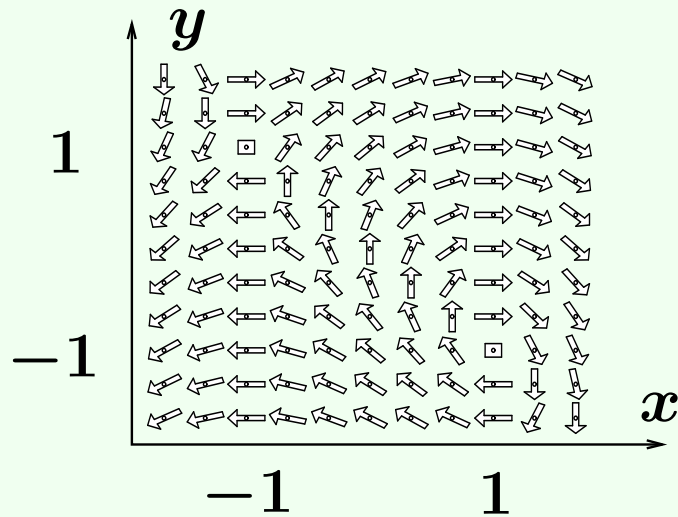
Figure 5. Equilibria $(1, -1)$, $(-1, 1)$ for (1).

Graph window $|x| \leq 2$, $|y| \leq 2$ with 11×11 uniform grid and full direction field. An arrow shaft at a grid point represents a solution curve over a small time interval. Threaded solution curves on long time intervals have tangents nearly matching nearby arrow shaft directions.
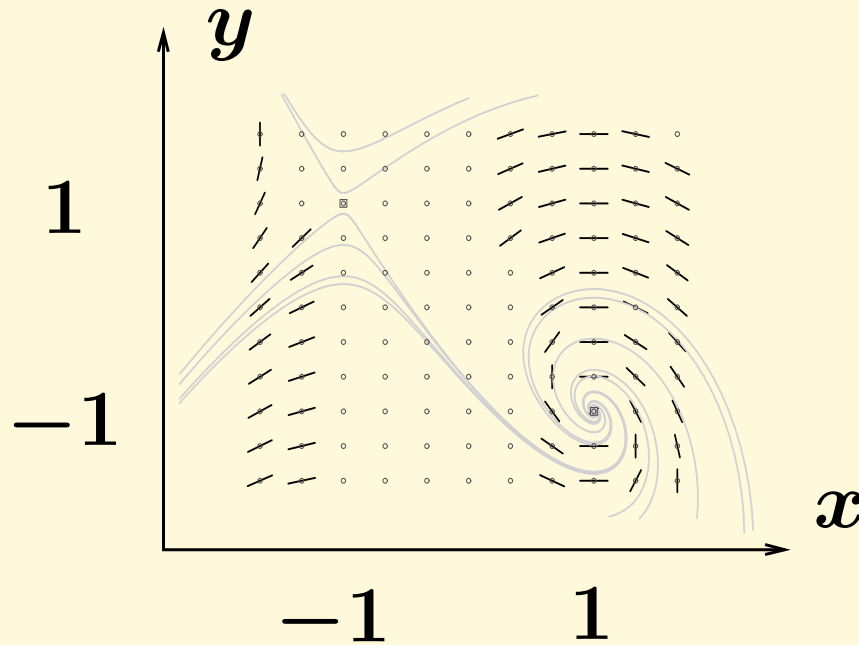
Figure 6. Equilibria $(1, -1)$, $(-1, 1)$ for (1).
Graph window with 11×11 uniform grid and partial direction field. Threaded solution curves have been added. Arrow shafts selected from certain direction field arrows. Threaded solution curve tangents nearly match nearby direction field arrow shafts. See Figure 1 for how to match tangents.
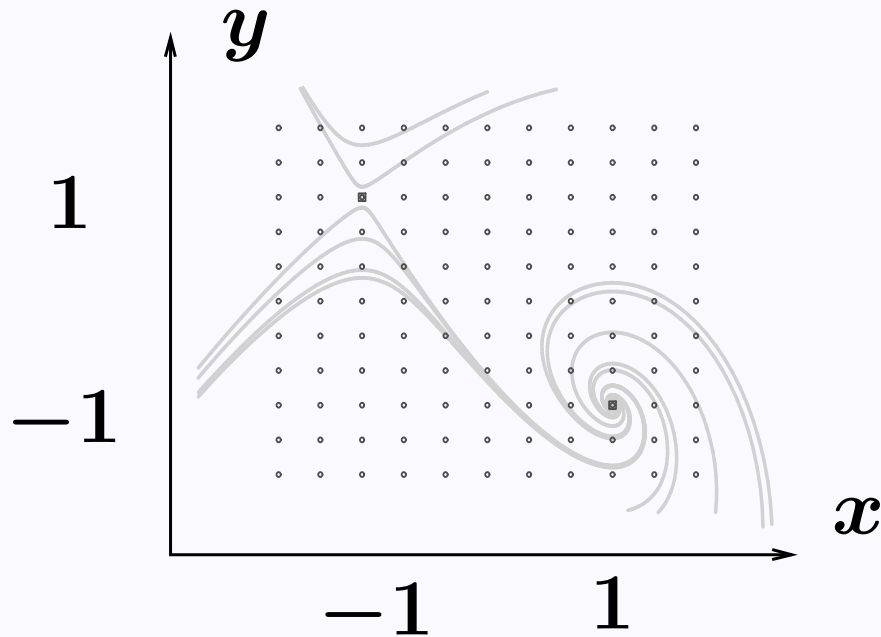
Figure 7. Phase portrait for (1).
The direction field has been removed for clarity. Shown are threaded solution curves and an 11×11 grid. Threaded solution curves do not actually cross, even though graphical resolution might suggest otherwise.

Phase plot by computer

Illustrated here is how to make a phase plot for the previous illustration using computer algebra system `maple`.

Before the computer work begins, the differential equation is defined and the equilibria are computed. Defaults supplied by `maple` allow an initial phase portrait to be plotted, from which the graph window is selected. The initial plot code:

```
with(DEtools):
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:
wind:=x=-2..2,y=-2..2:
DEplot({des},[x(t),y(t)],t=-20..20,wind);
```

Revised Computer Phase plot

The initial plot suggests which initial conditions near the equilibria should be selected in order to create typical orbits on the graphic. The final code with initial data and options:

```
with(DEtools):
 des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:
 wind:=x=-2..2,y=-2..2:
 opts:=stepsize=0.05,dirgrid=[13,13],
 axes=none,thickness=3,arrows=small:
 ics:=[[x(0)=-1,y(0)=1.1],[x(0)=-1,y(0)=1.5],
 [x(0)=-1,y(0)=.9],[x(0)=-1,y(0)=.6],[x(0)=-1,y(0)=.3],
 [x(0)=1,y(0)=-0.9],[x(0)=1,y(0)=-0.6],[x(0)=1,y(0)=-0.6],
 [x(0)=1,y(0)=-0.3],[x(0)=1,y(0)=-1.6],[x(0)=1,y(0)=-1.3],
 [x(0)=1,y(0)=-1.1]]:
 DEplot({des},[x(t),y(t)],t=-20..20,wind,ics,opts);
```

Since `maple` 12, a phase portrait tool exists which simplifies experimentation to draw threaded curves. The tool is recommended for phase plots that need little fine tuning.