Orthogonality

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Orthogonality

Definition 1 (Orthogonal Vectors)
Two vectors $\mathbf{u}$, $\mathbf{v}$ are said to be **orthogonal** provided their dot product is zero:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{0}.$$

If both vectors are nonzero (not required in the definition), then the angle $\boldsymbol{\theta}$ between the two vectors is determined by

$$\cos \boldsymbol{\theta} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \mathbf{0},$$

which implies $\boldsymbol{\theta} = 90^\circ$. In short, orthogonal vectors form a right angle.

Unitization

Any nonzero vector $\mathbf{u}$ can be multiplied by $c = \dfrac{1}{\|\mathbf{u}\|}$ to make a **unit vector** $\mathbf{v} = c\mathbf{u}$, that is, a vector satisfying $\|\mathbf{v}\| = 1$.

This process of changing the length of a vector to 1 by scalar multiplication is called **unitization**.

Orthogonal and Orthonormal Set

Definition 2 (Orthogonal Set of Vectors)
A given set of nonzero vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ that satisfies the **orthogonality condition**

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0, \quad i \neq j,$$

is called an **orthogonal set**.

Definition 3 (Orthonormal Set of Vectors)
A given set of unit vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ that satisfies the **orthogonality condition** is called an **orthonormal set**.

Independence and Orthogonality

Theorem 1 (Independence)
An orthogonal set of nonzero vectors is linearly independent.

Proof: Let $c_1, \ldots, c_k$ be constants such that nonzero orthogonal vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ satisfy the relation

$$c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

Take the dot product of this equation with vector $\mathbf{u}_j$ to obtain the scalar relation

$$c_1\mathbf{u}_1 \cdot \mathbf{u}_j + \cdots + c_k\mathbf{u}_k \cdot \mathbf{u}_j = 0.$$

Because all terms on the left are zero, except one, the relation reduces to the simpler equation

$$c_j\|\mathbf{u}_j\|^2 = 0.$$

This equation implies $c_j = 0$. Therefore, $c_1 = \cdots = c_k = 0$ and the vectors are proved independent.

Inner Product Spaces

An **inner product** on a vector space V is a function that maps a pair of vectors $\mathbf{u}$, $\mathbf{v}$ into a scalar $\langle \mathbf{u}, \mathbf{v} \rangle$ satisfying the following four properties.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [symmetry]

2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ [additivity]

3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ [homogeneity]

4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$ [positivity]

The **length** of a vector is then defined to be $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$.
A vector space V with inner product defined is called an **inner product space**.

Fundamental Inequalities

Theorem 2 (Cauchy-Schwartz Inequality)
In any inner product space V,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

Equality holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

Theorem 3 (Triangle Inequality)
In any inner product space V,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Pythagorean Relation

Theorem 4 (Pythagorean Identity)
In any inner product space V,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.