## Transform Properties

Collected here are the major theorems for the manipulation of Laplace transform tables.

- Lerch's Cancelation Law
- Linearity
- The Parts Rule ($t$-Derivative Rule)
- The $t$-Integral Rule
- The $s$-Differentiation Rule
- First Shifting Rule
- Second Shifting Rule
- Periodic Function Rule
- Convolution Rule

**Theorem 1 (Lerch)**
If $f_1(t)$ and $f_2(t)$ are continuous, of exponential order and

$$\int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty f_2(t)e^{-st}dt$$

for all $s > s_0$, then for $t \geq 0$,

$$f_1(t) = f_2(t).$$

The result is remembered as the cancelation law

$$L(f_1(t)) = L(f_2(t)) \text{ implies } f_1(t) = f_2(t).$$

**Theorem 2 (Linearity)**
The Laplace transform has these inherited integral properties:

$$\begin{array}{ll} \textbf{(a)} & L(f(t)+g(t)) = L(f(t)) + L(g(t)), \\ \textbf{(b)} & L(cf(t)) = cL(f(t)). \end{array}$$

**Theorem 3 (The Parts Rule)**
Let $y(t)$ be continuous, of exponential order and let $y'(t)$ be piecewise continuous on $t \geq 0$. Then $L(y'(t))$ exists and

$$L(y'(t)) = sL(y(t)) - y(0).$$

**Theorem 4 (The $t$-Integral Rule)**
Let $g(t)$ be of exponential order and continuous for $t \geq 0$. Then

$$L\left(\int_0^t g(x)\,dx\right) = \frac{1}{s}L(g(t)).$$

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- The parts rule is also called the $t$-derivative rule. It is used to remove derivatives $y'$ from Laplace equations.
- The two rules are related by $y(t) = \int_0^t g(x)dx$.

**Theorem 5 (The $s$-Differentiation Rule)**
Let $f(t)$ be of exponential order. Then

$$L(tf(t)) = -\frac{d}{ds}L(f(t)).$$

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The rule says that each factor of $(t)$ in the integrand of a Laplace integral can be crossed out provided an operation $-d/ds$ is inserted in front of the integral. It is remembered as

*multiplying by $(-t)$ differentiates the transform.*

**Theorem 6 (First Shifting Rule)**
Let $f(t)$ be of exponential order and $-\infty < a < \infty$. Then

$$L(e^{at} f(t)) = L(f(t))\big|_{s \to (s-a)}\,.$$

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The rule says that an exponential factor $e^{at}$ in the integrand can be crossed out, provided this action is compensated by replacing $s$ by $s - a$ in the answer. It is remembered as

*multiplying by $e^{at}$ shifts the transform $s \longrightarrow s - a$.*

## Heavisıde Step

The Step function is defined by $\text{step}(t) = 1$ for $t \geq 0$ and $\text{step}(t) = 0$ for $t < 0$. It is the same as the **unit step** $u(t)$ and the **Heaviside function** $H(t)$. Then $\text{step}(t-a)$ is the step function shifted from the origin to location $t = a$,

$$\text{step}(t-a) = \begin{cases} 1 & a \leq t < \infty, \\ \text{otherwise.} & \end{cases}$$

The function $\text{shelf}$ is is a finite interval step function defined by

$$\begin{aligned} \text{shelf}(t, a, b) &= \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} \\ &= \text{step}(t-a) - \text{step}(t-b). \end{aligned}$$

## Maple Worksheet Definitions

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step   := unapply(piecewise(t >= 0, 1, 0),t);
shelf  := unapply(step(t-a)-step(t-b),(t,a,b));
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## Step Function Shifting Rule

**Theorem 7 (Second Shifting Rule)**
Let $f(t)$ and $g(t)$ be of exponential order and assume $a \geq 0$. Let $u(t) = \mathrm{step}(t)$. Then

$$\begin{aligned} &\textbf{(a)} \quad L(f(t-a)u(t-a)) = e^{-as}L(f(t)), \\ &\textbf{(b)} \quad L(g(t)u(t-a)) = e^{-as}L(g(t+a)). \end{aligned}$$

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The relations are used to manipulate Laplace equations that arise in differential equations with piecewise defined inputs. Electrical engineering has many such examples.

**Theorem 8 (Periodic Function Rule)**
Let $f(t)$ be of exponential order and satisfy $f(t+P) = f(t)$. Then

$$L(f(t)) = \frac{\int_0^P f(t)e^{-st}dt}{1 - e^{-Ps}}.$$

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$$L(\textsf{floor}(t/a)) = \frac{e^{-as}}{s(1-e^{-as})}$$

Staircase function,
$\textsf{floor}(x) =$ greatest integer $\le x$.

$$L(\textsf{sqw}(t/a)) = \frac{1}{s}\tanh(as/2)$$

Square wave,
$\textsf{sqw}(x) = (-1)^{\textsf{floor}(x)}$.

$$L(a\,\textsf{trw}(t/a)) = \frac{1}{s^2}\tanh(as/2)$$

Triangular wave,
$\textsf{trw}(x) = \int_0^x \textsf{sqw}(r)dr$.

**Theorem 9 (Convolution Rule)**
Let $f(t)$ and $g(t)$ be of exponential order. Then

$$L(f(t))L(g(t)) = L\left(\int_0^t f(x)g(t-x)dx\right).$$

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An example:

$$\begin{aligned} \frac{1}{s^2}\ \frac{1}{s-2} &= L(t)L\left(e^{2t}\right) \\ &= L\left(\int_0^t xe^{2(t-x)}dx\right) \\ &= L\left(\frac{1}{3}e^{2t} - \frac{1}{2}t - \frac{1}{4}\right) \end{aligned}$$