Elementary Matrices and Frame Sequences

- Elementary Matrices
- Fundamental Theorem on Elementary Matrices
- A certain 6-frame sequence.
- Frame Sequence Details
- Summary

Elementary Matrices

An elementary matrix $\boldsymbol{E}$ is the result of applying a combination, multiply or swap rule to the identity matrix. The computer algebra system `maple` displays typical 4×4 elementary matrices (C=Combination, M=Multiply, S=Swap) as follows.

```
with(linalg):                 | with(LinearAlgebra):
Id:=diag(1,1,1,1);            | Id:=IdentityMatrix(4);
C:=addrow(Id,2,3,c);          | C:=RowOperation(Id,[3,2],c);
M:=mulrow(Id,3,m);            | M:=RowOperation(Id,3,m);
S:=swaprow(Id,1,4);           | S:=RowOperation(Id,[4,1]);
```

The answers:

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \boldsymbol{c} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boldsymbol{m} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{S} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Constructing elementary matrices E

Mult	Change a one in the identity matrix to symbol $m \neq 0$.
Combo	Change a zero in the identity matrix to symbol c.
Swap	Interchange two rows of the identity matrix.

Constructing E^{-1} from elementary matrix E

Mult	Change diagonal multiplier $m \neq 0$ in E to $1/m$.
Combo	Change multiplier c in E to $-c$.
Swap	The inverse of E is E itself.

Fundamental Theorem on Elementary Matrices

Theorem 1 (The rref and elementary matrices)
Let $\boldsymbol{A}$ be a given matrix of row dimension $\boldsymbol{n}$. Then there exist $\boldsymbol{n} \times \boldsymbol{n}$ elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ such that

$$\mathbf{rref}(\boldsymbol{A}) = \boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}.$$

Details of Proof

The result is the observation that left multiplication of matrix $\boldsymbol{A}$ by elementary matrix $\boldsymbol{E}$ gives the answer $\boldsymbol{EA}$ for the corresponding multiply, combination or swap operation. The matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots$ represent the multiply, combination and swap operations performed in the frame sequence which take the First Frame into the Last Frame, or equivalently, original matrix $\boldsymbol{A}$ into $\mathbf{rref}(\boldsymbol{A})$.

A certain 6-frame sequence ________________________________.

$$A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 6 & 3 \end{pmatrix}$$ Frame 1, original matrix.

$$A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 3 & 6 & 3 \end{pmatrix}$$ Frame 2, combo(1,2,-2).

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 3 \end{pmatrix}$$ Frame 3, mult(2,-1/6).

$$A_4 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{pmatrix}$$ Frame 4, combo(1,3,-3).

$$A_5 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ Frame 5, combo(2,3,-6).

$$A_6 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ Frame 6, combo(2,1,-3). Found $\mathbf{rref}(A_1)$.

Continued

The corresponding 3×3 elementary matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 2, combo(1,2,-2) applied to I.

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 3, mult(2,-1/6) applied to I.

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$ Frame 4, combo(1,3,-3) applied to I.

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix}$$ Frame 5, combo(2,3,-6) applied to I.

$$E_5 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 6, combo(2,1,-3) applied to I.

Frame Sequence Details

$A_2 = E_1 A_1$	Frame 2, E_1 equals combo(1,2,-2) on I.
$A_3 = E_2 A_2$	Frame 3, E_2 equals mult(2,-1/6) on I.
$A_4 = E_3 A_3$	Frame 4, E_3 equals combo(1,3,-3) on I.
$A_5 = E_4 A_4$	Frame 5, E_4 equals combo(2,3,-6) on I.
$A_6 = E_5 A_5$	Frame 6, E_5 equals combo(2,1,-3) on I.
$A_6 = E_5 E_4 E_3 E_2 E_1 A_1$	Summary frames 1-6.

Then

$$\mathbf{rref}(A_1) = E_5 E_4 E_3 E_2 E_1 A_1,$$

which is the result of the Theorem.

Summary

The summary:

$$\boldsymbol{A_6} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{A_1}$$

Because $\boldsymbol{A_6} = \mathbf{rref}(\boldsymbol{A_1})$, the above equation gives the inverse relationship

$$\boldsymbol{A_1} = \boldsymbol{E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}}\, \mathbf{rref}(\boldsymbol{A_1}).$$

Each inverse matrix is simplified by the rules for constructing $\boldsymbol{E^{-1}}$ from elementary matrix $\boldsymbol{E}$, the result being

$$\boldsymbol{A_1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{rref}(\boldsymbol{A_1})$$