Solution Set Basis for Linear Differential Equations

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Linear Differential Equations

The solution set of a homogeneous constant coefficient linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = 0$$

is known to be a vector space of functions of dimension n, consisting of special linear combinations

(1)
$$y = c_1 f_1 + \cdots + c_n f_n,$$

where f_1, \ldots, f_n are elementary functions known as **atoms**.

Definition of Atom

An atom is defined as the nonzero real or imaginary part of

```
x^n e^{ax} (\cos bx + i \sin bx)
```

with a, b real, $b \ge 0$, and n = 0, 1, 2... an integer. An atom has coefficient 1, and the zero function is not an atom.

Examples of Atoms

1, x, x^2 , e^x , xe^{-x} , $x^{15}e^{2x}\cos 3x$, $\cos 3x$, $\sin 2x$, $x^2\cos 2x$, $x^6\sin \pi x$, $x^{10}e^{\pi x}\sin 0.1x$

Functions that are not Atoms

 $x/(1+x), \ln |x|, e^{x^2}, \sin(x+1), 0, 2x, \sin(1/x), \sqrt{x}$

Theorems about Atoms

Theorem 1 (Independence)

Any finite list of atoms is linearly independent.

Theorem 2 (Euler)

The *real* characteristic polynomial $p(r)=r^n+a_{n-1}r^{n-1}+\dots+a_0$ has a factor $(r-a-ib)^{k+1}$ if and only if

$$y = x^k e^{ax} (\cos bx + i \sin bx)$$

is a solution of (1).

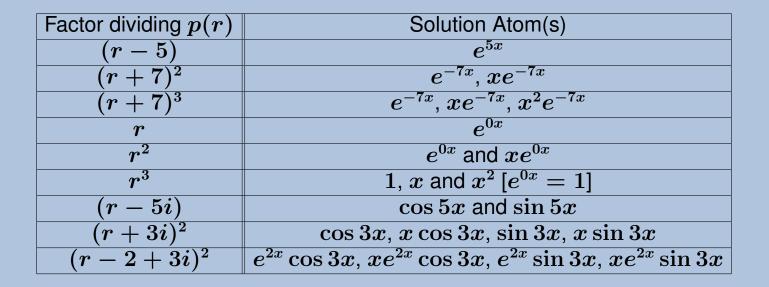
Theorem 3 (Real Solutions)

If u and v are real and u + iv is a solution of equation (1), then u and v are real solutions of equation (1).

Theorem 4 (Basis)

The solution set of equation (1) has a basis of n solution atoms which are determined by Euler's theorem.

Euler's Theorem Translated _____ Theorem 5 (How to Apply Euler's Theorem)



Example 1. Solve y''' = 0. **Solution**: $p(r) = r^3$ implies 1, x, x^2 are solution atoms. They are independent, hence form a basis for the 3-dimensional solution space. Then $y = c_1 + c_2 x + c_3 x^2$.

Example 2. Solve y'' + 4y = 0. **Solution:** $p(r) = r^2 + 4$ implies $\cos 2x$ and $\sin 2x$ are solution atoms. They are a basis for the 2-dimensional solution space with $y = c_1 \cos 2x + c_2 \sin 2x$.

Example 3. Solve y'' + 2y' = 0. **Solution**: $p(r) = r^2 + 2r$ implies 1, e^{-2x} are solution atoms. These independent atoms form a basis. Then $y = c_1 + c_2 e^{-2x}$.

Example 4. Solve $y^{(4)} + 4y'' = 0$. Solution: $p(r) = r^4 + 4r^2 = r^2(r^2 + 4)$ implies the four atoms 1, x, cos 2x, sin 2x are solutions. Then $y = c_1 + c_2x + c_3 \cos 2x + c_3 \sin 2x$.

Example 5. Solve the differential equation if $p(r) = (r^3 - r^2)(r^2 - 1)(r^2 + 4)^2$. **Solution**: The distinct factors of p(r) are r^2 , $(r - 1)^2$, r + 1, $(r - 2i)^2$, $(r + 2i)^2$. Euler's theorem implies the DE has nine solution atoms 1, x, e^x , xe^x , e^{-x} , $\cos 2x$, $x \cos 2x$, $\sin 2x$, $x \sin 2x$. Then y is a linear combination of these atoms.