How to Solve Linear Differential Equations

- Atoms
- Independence of Atoms
- Construction of the General Solution from a List of Distinct Atoms
- Euler's Theorem
- The Atom List and Euler's Method
- Explanation of Euler's Method
- Main Theorems on Atoms and Linear Differential Equations

Atoms

An **atom** is a term with coefficient 1 obtained by taking the real and imaginary parts of

$$x^j e^{ax}(\cos cx + i \sin cx), \quad j = 0, 1, 2, \ldots,$$

where a and c represent real numbers and $c \geq 0$. By definition, zero is not an atom.

Details and Remarks

- Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies that an atom is constructed from the expression $x^j e^{zx}$ where $z = a + ic$.
- An atom is a term of one of the following types:

  $$x^n, \; x^n e^{ax}, \; x^n e^{ax} \cos bx, \; x^n e^{ax} \sin bx.$$

  The symbol n is an integer $0, 1, 2, \ldots$ and a, b are real numbers with $b > 0$.
- In particular, the powers $1, x, x^2, \ldots, x^k$ are atoms.
- The term that makes up an atom has coefficient 1, therefore $2e^x$ is not an atom, but the 2 can be stripped off to create the atom e^x. Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms x and x^2 are indeed atoms. Terms like e^{x^2}, $\ln|x|$ and $x/(1 + x^2)$ are not atoms, nor are they constructed from atoms.

Independence

Linear algebra defines a list of functions $f_1, \ldots, f_k$ to be **linearly independent** if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) \quad \text{for all } x$$

implies $c_1 = c_2 = \cdots = c_k = 0$.

Independence and Atoms

Theorem 1 (Atoms are Independent)
A list of finitely many distinct atoms is linearly independent.

Theorem 2 (Powers are Independent)
The list of distinct atoms $1, x, x^2, \ldots, x^k$ is linearly independent. And all of its sublists are linearly independent.

Construction of the General Solution from a List of Distinct Atoms

- **Picard's theorem** says that the homogeneous constant-coefficient linear differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

  has solution space S of dimension n. Picard's theorem reduces the general solution problem to finding n linearly independent solutions.

- **Euler's theorem** *infra* says that the required n independent solutions can be taken to be atoms. The theorem explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation [**characteristic polynomial**=left side]

$$r^n + p_{n-1}r^{n-1} + \cdots + p_1 r + p_0 = 0.$$

- The **Fundamental Theorem of Algebra** states that there are exactly n roots r, real or complex, for an nth order polynomial equation. The result implies that the characteristic equation has exactly n roots, counting multiplicities.

- **General Solution**. Because the list of atoms constructed by Euler's theorem has n distinct elements, then this list of independent atoms forms a **basis** for the general solution of the differential equation, giving

$$y = c_1(\text{atom } 1) + \cdots + c_n(\text{atom } n).$$

  Symbols $c_1, \ldots, c_n$ are arbitrary coefficients. In particular, each atom listed is itself a solution of the differential equation.

Euler's Theorem

Theorem 3 (L. Euler)
The function $y = x^j e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential of the nth order if and only if $(r - r_1)^{j+1}$ divides the characteristic polynomial.

The Atom List

1. If r_1 is a real root, then the atom list for r_1 begins with $e^{r_1 x}$. The revised atom list is

$$e^{r_1 x}, xe^{r_1 x}, \ldots, x^{k-1}e^{r_1 x}$$

provided r_1 is a root of multiplicity k. This means that factor $(r - r_1)^k$ divides the characteristic polynomial, but factor $(r - r_1)^{k+1}$ does not.

2. If $r_1 = \alpha + i\beta$, with $\beta > 0$ and its conjugate $r_2 = \alpha - i\beta$ are roots of the characteristic equation, then the atom list for this pair of roots (both r_1 and r_2 counted) begins with

$$e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x.$$

For a root of multiplicity k, these real atoms are multiplied by atoms $1, \ldots, x^{k-1}$ to obtain a list of $2k$ atoms

$$\begin{array}{llll} e^{\alpha x}\cos\beta x, & xe^{\alpha x}\cos\beta x, & \ldots, & x^{k-1}e^{\alpha x}\cos\beta x, \\ e^{\alpha x}\sin\beta x, & xe^{\alpha x}\sin\beta x, & \ldots, & x^{k-1}e^{\alpha x}\sin\beta x. \end{array}$$

Explanation of steps 1 and 2

1. Root r_1 always produces atom e^{r_1x}, but if the multiplicity is $k > 1$, then e^{r_1x} is multiplied by the list of atoms $1, x, \ldots, x^{k-1}$.

2. The expected first terms e^{r_1x} and e^{r_2x} [$e^{\alpha x+i\beta x}$ and $e^{\alpha x-i\beta x}$] are **not atoms**, but they are **linear combinations of atoms**:

$$e^{\alpha x\pm i\beta x} = e^{\alpha x}\cos\beta x \pm i e^{\alpha x}\sin\beta x.$$

The atom list for a complex conjugate pair of roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$ is obtained by multiplying the two *real* atoms

$$e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x$$

by the powers

$$1, x, \ldots, x^{k-1}$$

to obtain the $2k$ distinct *real* atoms in item **2** above.

Theorem 4 (Homogeneous Solution y_h and Atoms)
Linear homogeneous differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of atoms.

Theorem 5 (Particular Solution y_p and Atoms)
A linear non-homogeneous differential equation with constant coefficients a having forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

Theorem 6 (General Solution y and Atoms)
A linear non-homogeneous differential equation with constant coefficients having forcing term

$$f(x) = \text{ a linear combination of atoms}$$

has general solution

$$y(x) = y_h(x) + y_p(x) = \text{ a linear combination of atoms.}$$

Proofs

The first theorem follows from Picard's theorem, Euler's theorem and independence of atoms. The second follows from the method of undetermined coefficients, *infra*. The third theorem follows from the first two.