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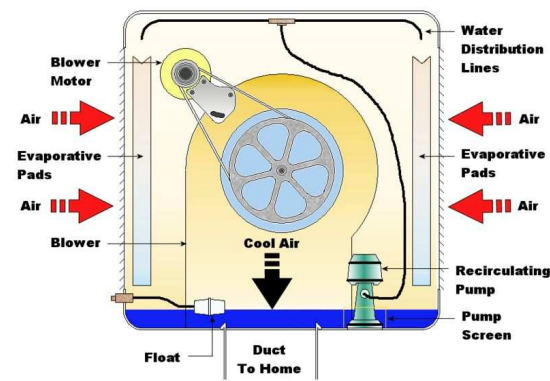
Math 2250 Maple Project 2: Newton Cooling
January 2009

Due date: See the internet due dates.

References: Edwards-Penney 2nd, pages 55–57. Code in `maple` appears in `2250mapleL2-F2008.txt` at URL
`http://www.math.utah.edu/~gustafso/`. This document: `2250mapleL2-F2008.pdf`.

An evaporative cooler, used in low-humidity regions of the United States instead of refrigerated air conditioning.

The Problem and Model. The project is to determine an approximation for the indoor temperature $u(t)$ in a building equipped with an evaporative cooler. The model uses Newton's cooling law, insulation data k_1, cooler data k_2, symbol $k = k_1 + k_2$, a formula for the ambient outside temperature $A(t)$ and a formula for the cooler air temperature $C(t)$ (see the background section *infra*):

(1)
$$u'(t) + ku(t) = k_1 A(t) + k_2 C(t),$$
$$u(0) = u_0.$$

Assumptions and Notation.

- The inside temperature is the wall thermometer reading $u(t)$, where t is in hours measured from midnight $t = 0$.

- Symbol u_0 is the wall thermometer reading at midnight.

- The model assumes that no heat sources or heat sinks are present inside the building, except for the evaporative cooler.

- The ambient outside air temperature is called $A(t)$. It varies from a low of $m = 60$ degrees to a high of $M = 98$ degrees with the low and high at 3am and 3pm, respectively. Let $\omega_0 = \pi/12$. The ambient temperature model will be

$$\begin{aligned} A(t) &= \frac{1}{2}(M + m) - \frac{1}{2}(M - m)\cos\omega_0(t - 3) \\ &= 79 - 19\cos\omega_0(t - 3). \end{aligned}$$

Then $A(t)$ satisfies $M = \max A(t) = A(15) = 98$, $m = \min A(t) = A(3) = 60$ and $A(t)$ is t-periodic of period 24 hours. For use in `maple`, define a function of two variables

<pre><code>AA:=unapply(79-19*cos (omega* (t-3)),(t,omega));</code></pre>

Maple function `AA(t,omega)` is useful to maintain the symbol ω in displays.

- The building is assumed insulated, the average insulation factor being represented by a Newton cooling constant k_1, usually $0.2 \leq k_1 \leq 0.5$. The expected temperature change due to insulation is $k_1(A(t) - u(t))$.

- Let $C(t)$ be the evaporative cooler air temperature. During operation, the evaporative cooler is assumed to deliver air 20% cooler than the ambient temperature, which becomes the assumption $C(t) = 0.8A(t)$. The lower temperature is due to heat loss from water evaporation. The expected temperature change due to cooler operation is $k_2(C(t) - u(t))$, where k_2 is a Newton cooling constant.

- Symbol $k = k_1 + k_2$. Symbol $k_3 = k_1 + 0.8k_2$.

- Symbol $u_h(t) = ce^{-kt}$ denotes the general solution of the homogeneous equation $u' + ku = 0$.

- Symbol $u_p(t)$ stands for a particular solution of $u' + ku = k_3 A(t)$.

Problem L2.1. (Solution Formulas for u_p and u)

Derive by hand, using $\texttt{maple}$ assist for values of integrals, an explicit symbolic formula for $u_p(t)$. Display a final formula for $u = u_h + u_p$ which depends only on t, u_0, k, k_3 and ω. Check your hand answer for u in $\texttt{maple}$. The only $\texttt{maple}$ assist in this problem is $\texttt{maple}$ integration and the $\texttt{maple}$ answer check.

Problem L2.2. (Steady-State Periodic Solution)

Derive by hand a formula for the steady-state periodic solution u_{SS} of $u' + ku = k_3 A(t)$. There is no $\texttt{maple}$ assist in this problem, it is all done by hand from the answer in L2.1. The results:

$$
\begin{aligned}
u_{\text{SS}}(t) &= \frac{k_3}{k}\left(79 - \frac{19}{k^2 + \omega_0^2}(k\cos\omega_0(t-3) + \omega_0\sin\omega_0(t-3))\right), \\
u(t) &= Ce^{-kt} + u_{\text{SS}}(t), \\
C &= u_0 - u_{\text{SS}}(0).
\end{aligned}
$$

Problem L2.3. (Indoor-Outdoor Variation)

Compare in a $\texttt{maple}$ graphic oscillations of the indoor temperature $u(t)$ and the outdoor temperature $A(t)$ over a 48-hour period assuming $k_1 = 0.32$, $k_2 = 0.5$, $u_0 = 76$, $\omega = \pi/12$. Include in the plot four horizontal lines for the maximum and minimum temperatures for each of $u(t)$ and $A(t)$. Report the indoor and outdoor temperature variations from this 48-hour plot (max minus min in each case). Report the phase delay between the indoor and outdoor temperatures, using this same plot.

Problem L2.4. (Swamp Cooler)

Assume $\omega = \pi/12$ and $k_2 = 0.5$. Consider all insulation constants k_1 in the range from 0.2 to 0.48. Suppose the inside temperature is 76 degrees at midnight. Report approximate ranges of hours and insulation constant k_1, during the first 72 hours, for which the indoor temperature is at or below 70 degrees. Justify your logic used to find the ranges, in a short paragraph. Illustrate with a computer graphic.

Staple this page on top of all hand and maple work.
Background, problem notes and sample maple code are on the following pages . . .

Background

Newton Cooling Model. Newton's law of cooling is:

> *The rate of change of a body's temperature is proportional to the difference between the ambient temperature and the body's temperature.*

This implies that du/dt is the sum of two rates, one due to the insulation and one due to the evaporative cooler. A model for the indoor temperature $u(t)$ is given by $du/dt = k_1(A(t) - u(t)) + k_2(C(t) - u(t))$, $u(0) = u_0$, which can be rearranged as

$$(2) \qquad u'(t) + ku(t) = k_3A(t), \quad k = k_1 + k_2, \quad k_3 = k_1 + 0.8k_2, \quad u(0) = u_0.$$

Indoor Temperature $u(t)$. The integrating factor method for linear equations applies to find the general solution by these steps:

$u' + ku = k_3A(t)$	Copy the differential equation. The integrating factor is $W = e^{kt}$.
$\dfrac{(e^{kt}u)'}{e^{kt}} = k_3A(t)$	Integrating factor method: replace $u' + ku$ by $(Wu)'/W$.
$(e^{kt}u)' = k_3A(t)e^{kt}$	Clear fractions.
$e^{kt}u = u_0 + \int_0^t k_3A(x)e^{kx}\,dx$	Integrate both sides with respect to t. Apply the Fundamental Theorem of Calculus. Use $u(0) = u_0$.
$u = u_0e^{-kt} + e^{-kt}\int_0^t k_3A(x)e^{kx}\,dx$	Divide to isolate the *explicit solution u*.

Let $u_h(t) = u_0e^{-kt}$, which is a solution of $u' + ku = 0$. Let $u_p(t) = k_3e^{-kt}\int_0^t e^{kx}A(x)dx$, a particular solution of the nonhomogeneous differential equation $u' + ku = k_3A(t)$. Then the indoor temperature $u = u_h + u_p$ depends on the time t, the initial temperature u_0, the insulation constant k_1, the evaporative cooler constant k_2 and the frequency ω (which is fixed at $\pi/12$). Because $k = k_1 + k_2$ and $k_3 = k_1 + 0.8 * k_2$, then write $u = u(t, u_0, k, k_3, \omega)$ to emphasize the dependence. In `maple`, advantages exist for adding the variable name ω, which is later set equal to value $\omega_0 = \pi/12$. Write u as $\boxed{\texttt{U(t,u0,k,k3,omega)}}$ for use in `maple`.

Steady-state solution. The steady-state solution u_{SS} is an expression obtained from the general solution formula $u = u_h + u_p$ by dropping all terms containing a negative exponential. It depends on t, k, k_3 and ω [but not u_0]. This expression is known to be a solution of the differential equation, independent of any arbitrary constants, which represents the **observed temperature** after a long time. In short, u_{SS} is independent of u_0. Theory provides the following result.

> **Theorem** (Unique Periodic Solution) Assume $p \neq 0$ is a constant and $q(x)$ is continuous and T-periodic $[q(x + T) = q(x)]$. Then the differential equation $\dfrac{dy}{dx} + py = q(x)$ has a unique T-periodic solution $y(x)$ given by the initial value
> $$y(0) = \left(e^{pT} - 1\right)^{-1} \int_0^T e^{pt}q(t)dt.$$

Problem Notes

Notes on L2.1

A derivation by hand is expected, which parallels the derivation above. The difference is the inclusion of fine details, including integration results, finally giving a formula for u_h and a formula for u_p. Integration for the u_p formula is referenced to an attached maple worksheet appendix. The answer check references a second attached worksheet.

Because many students want to know how to do the integration, based upon a calculus background, more information is provided here. The integration problem can be solved by hand using the book's integral table:

$$u_p(t) = k_3 e^{-kt} \int_0^t e^{kx} \left(79 - 19\cos\left(\omega(x-3)\right)\right) dx.$$

A change of variable $u = x - 3$ allows use of integral table entry $\int e^{au} \cos bu \, du$. The symbol ω will be set to $\pi/12$, but for simplicity, use symbol ω throughout.

You are asked to use `maple` to evaluate the integral. Its table lookup method is superior to the hand method above, in that no change of variable is required to obtain the answer. This method is accurate and avoids primary pitfalls in using a book table. A coding example appears below, which moves the factor $k_3 e^{-kt}$ inside the integrand, in order to automatically perform simplifications.

```
# maple integral table lookup
unassign('omega','t','u0','k','k1','k2','k3','x');
integrand:=(79-19*cos(omega*(x-3)))*k3*exp(k*x-k*t);
integral:=int(integrand,x=0..t);
```

The formulas for u_h and u_p are used again in L2.2 in order to derive the steady-state solution. Your answer for $u = u_h + u_p$ *must contain* symbols $\boxed{\texttt{t, u0, k, k3, omega}}$.

The answer check in `maple` is organized as follows. The complications of setting $\omega = \pi/12$ are avoided here by leaving ω as a symbol, since it does not affect the answer check.

```
# Test LHS=RHS for u'+ku=k3 A.
unassign('t','C','omega','k','k1','k2','k3'):
k:=k1+k2:k3:=k1+0.8*k2:
uh:=C*exp(-k*t):
up:=(explicit answer from L2.1):
myANS:=uh+up:
LHS:=diff(myANS,t)+k*myANS:
RHS:=k3*(79-19*cos(omega*(t-3))):
simplify(expand(LHS-RHS));
```

A successful test of $LHS = RHS$ produces answer *zero*, or an expression that reduces to zero.

Notes on L2.2

The steady-state solution is derived from the hand-generated symbolic solution $u = u_h + u_p$ in L2.1 by dropping all terms that contain the negative exponential factor e^{-kt}. The answer, where $\omega_0 = \pi/12$:

$$u_{\text{SS}} = \frac{k_3}{k}\left(79 - \frac{19}{k^2 + \omega_0^2}\left(k\cos\omega_0(t-3) + \omega_0\sin\omega_0(t-3)\right)\right).$$

To get `maple` to report the above formula, it is essential to evaluate everything with ω as a *symbol*, to wit, use in `maple` the statement $\boxed{\texttt{unassign('omega'):}}$. **Beware** of writing $\boxed{\texttt{omega:=Pi/12}}$, which defines ω to be a constant, unless you undo the effect immediately afterward. Maple's $\boxed{\texttt{unassign}}$ can be used to erase assignments.

Maple is unable to evaluate $\boxed{\texttt{limit(exp(-k*t),t=infinity)}}$ as zero, because the expression contains k, an unevaluated symbol. This is why you are asked to report the steady-state solution manually, by dropping terms from previously obtained answers.

Notes on L2.3

The outside temperature $A(t)$ (`AA(t,omega)` in `maple`) and the solution $u = U(t, u_0, k, k3, \omega)$ obtained in L2.2 can be coded in `maple` as follows:

```
unassign('t','u0','k','k1','k2','k3','omega'):
AA:=unapply(79-19*cos(omega*(t-3)),(t,omega));
uss := unapply(k3*(79*k^2+79*omega^2-19*k^2*cos(omega*(t-3))-
19*k*omega*sin(omega*(t-3)))/(k*(k^2+omega^2)),(t, k, k3, omega));
uss0:=subs(t=0,uss);
U:=unapply((u0-uss0)*exp(-k*t)+uss,(t,u0,k,k3,omega)):
```

The two curves plus a horizontal line $u = 70$ are placed onto the **plot** by this `maple` command:

```
plot({U(t,76,0.32,0.5,Pi/12),AA(t,0.5,Pi/12),70},t=0..48);
```

Missing curves? Probably, the missing curves are defined to contain an unevaluated variable name, like $\boxed{\texttt{pi}}$ instead of $\boxed{\texttt{Pi}}$. While `Pi` is the constant 3.14159, lowercase symbol `pi` is a variable name: *case* is significant in `maple`.

Curves vanish on the printer? Add $\boxed{\texttt{color=black}}$ to the plot command, e.g., $\boxed{\texttt{plot(sin(x),x=0..Pi,color=black);}}$.

Click a mouse button on the high and low spots in the graphic. Somewhere on the `maple` worksheet the coordinates of the click are displayed (look around on the screen!). This method gives decimal approximations to the max and min values. Use the four values obtained to expand the list of horizontal lines to four lines. The final plot has six curves.

The **indoor temperature variation** is just the maximum of u minus the minimum of u, as computed from the plot of $u(t)$. The outdoor temperature variation is computed from the graph of $A(t)$, in the same way.

The **phase shift** between the indoor and outdoor temperatures is computed as $|T_2 - T_1|$, where $A(T_1) = \max A(t)$ and $u(T_2) = \max u(t)$. Look at the graphic to find sane answers for T_1 and T_2. See the textbook for a more complete discussion of the ideas. Beware: T_1 and T_2 are values for time in the first 72 hours (abscissa values, not ordinate values). The smallest phase shift obtainable from the graphic is to be reported, because the transient solution Ce^{-kt} may affect the graph of $u(t) = Ce^{-kt} + u_{\mathrm{SS}}(t)$ for a few hours.

Notes on L2.4

A computer algebra assist for this problem uses `maple`'s function `implicitplot`. This function can plot the equation $u(t, 76, k, k3, \pi/12) = 70$ over the domain $0 \le t \le 72$, $0.2 \le k_1 \le 0.48$, $k = k_1 + k_2$, $k_3 = k_1 + 0.8k_2$, $k_2 = 0.5$. From this plot, and the 3D-plot $z = u(x, 76, y, k3, \pi/12)$, the question is easily answered.

```
with(plots): unassign('t','u0','k','k3','omega'):
uh:=u0*exp(-k*t):
up:=(the answer from L2.1):
myANS:=uh+up:
U:=unapply(myANS,(t,u0,k,k3,omega));k2:=0.5:
implicitplot(U(t,76,k1+k2,k1+0.8*k2,Pi/12)=70,t=0..72,k1=0.2..0.48);
plot3d({U(t,76,k1+k2,k1+0.8*k2,Pi/12),70},t=0..72,k1=0.2..0.48);
```

The relation between the 2D implicit plot and the 3D plot is seen by slicing the 3D plot at height $z = 70$ to obtain a *bread slice* parallel to the xy-plane. The implicit plot depicts the bread slice by projection onto the xy-plane. The slice can be visualized in `maple` by grabbing the 3D graphic with the mouse, to perform a rotation, which exposes a view of the underside of the 3D figure. The normal 3D view has the axes reversed, but mouse tools can be used to orient the 3D graphic to match the 2D bread slice.

Zoom in on the implicit plot by using a smaller time domain, suggested by the larger plot. The zoom should narrow in on a range of k-values and t-values such that the 3D surface is entirely below the plane $z = 70$. This geometry detects sub-70°F in the given range.

Physically, inside temperature 70 degrees or less is reached about a hour *after* the outside temperature drops low enough to generate evaporative cooler draft temperatures below 70 degrees. During 72 hours, there are four such inside temperature drops, verified from the 3D-plot, where z=temperature. Please report each of the four answers in this format: Sub-70°F from 0.75 to 10.2 hours, all k. An answer is correct if it is supported by the graphic.