What's Eigenanalysis?

Matrix eigenanalysis is a computational theory for the matrix equation $\mathbf{y} = \boldsymbol{A}\mathbf{x}$. For exposition purposes, we assume $\boldsymbol{A}$ is a $\mathbf{3} \times \mathbf{3}$ matrix.

Fourier's Eigenanalysis Model

$$
\begin{aligned}
\mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \text{ implies} \\
\mathbf{y} &= \boldsymbol{A}\mathbf{x} \\
&= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.
\end{aligned} \tag{1}
$$

The scale factors λ_1, λ_2, λ_3 and independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ depend only on $\boldsymbol{A}$. Symbols c_1, c_2, c_3 stand for arbitrary numbers. This implies variable $\mathbf{x}$ exhausts all possible $\mathbf{3}$-vectors in $\boldsymbol{R}^3$.

Fourier's model is a replacement process

$$A\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3\right) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

To compute $A\mathbf{x}$ from $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, replace each vector $\mathbf{v}_i$ by its scaled version $\lambda_i\mathbf{v}_i$.

Fourier's model is said to **hold** provided there exist scale factors and independent vectors satisfying (1). Fourier's model is known to fail for certain matrices A.

Powers and Fourier's Model

Equation (1) applies to compute powers $\boldsymbol{A}^n$ of a matrix $\boldsymbol{A}$ using only the basic vector space toolkit. To illustrate, only the vector toolkit for $\boldsymbol{R}^3$ is used in computing

$$\boldsymbol{A}^5\mathbf{x} = \boldsymbol{x}_1\boldsymbol{\lambda}_1^5\mathbf{v}_1 + \boldsymbol{x}_2\boldsymbol{\lambda}_2^5\mathbf{v}_2 + \boldsymbol{x}_3\boldsymbol{\lambda}_3^5\mathbf{v}_3.$$

This calculation does not depend upon finding previous powers $\boldsymbol{A}^2$, $\boldsymbol{A}^3$, $\boldsymbol{A}^4$ as would be the case by using matrix multiply.

Differential Equations and Fourier's Model

Systems of differential equations can be solved using Fourier's model, giving a compact and elegant formula for the general solution. An example:

$$\begin{aligned} x_1' &= x_1 + 3x_2, \\ x_2' &= \phantom{x_1 + {}} 2x_2 - x_3, \\ x_3' &= - 5x_3. \end{aligned}$$

The general solution is given by the formula [Fourier's theorem, proved later]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-5t} \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix},$$

which is related to Fourier's model by the symbolic formula

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3.$$

Fourier's model illustrated

Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = -5,$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.$$

Then Fourier's model holds (details later) and

$$\begin{aligned} \mathbf{x} &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} \quad \text{implies} \\ \boldsymbol{A}\mathbf{x} &= c_1(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} \end{aligned}$$

Eigenanalysis might be called *the method of simplifying coordinates*. The nomenclature is justified, because Fourier's model computes $\mathbf{y} = \boldsymbol{A}\mathbf{x}$ by scaling independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, which is a triad or **coordinate system**.

What is Eigenanalysis?

The subject of **eigenanalysis** discovers a coordinate system and scale factors such that Fourier's model holds. Fourier's model simplifies the matrix equation $\mathbf{y} = \boldsymbol{A}\mathbf{x}$, through the formula

$$\boldsymbol{A}(\boldsymbol{c}_1\mathbf{v}_1 + \boldsymbol{c}_2\mathbf{v}_2 + \boldsymbol{c}_3\mathbf{v}_3) = \boldsymbol{c}_1\boldsymbol{\lambda}_1\mathbf{v}_1 + \boldsymbol{c}_2\boldsymbol{\lambda}_2\mathbf{v}_2 + \boldsymbol{c}_3\boldsymbol{\lambda}_3\mathbf{v}_3.$$

What's an Eigenvalue?

It is a scale factor. An eigenvalue is also called a *proper value* or a *hidden value*. Symbols $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, $\boldsymbol{\lambda}_3$ used in Fourier's model are eigenvalues.

What's an Eigenvector?

Symbols $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in Fourier's model are called eigenvectors, or *proper vectors* or *hidden vectors*. They are assumed independent.

> The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues).

A Key Example

Let x in R^3 be a data set variable with coordinates x_1, x_2, x_3 recorded respectively in units of meters, millimeters and centimeters. We consider the problem of conversion of the mixed-unit x-data into proper MKS units (meters-kilogram-second) y-data via the equations

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= 0.001 x_2, \\ y_3 &= 0.01 x_3. \end{aligned} \tag{2}$$

Equations (2) are a **model** for changing units. Scaling factors $\lambda_1 = 1$, $\lambda_2 = 0.001$, $\lambda_3 = 0.01$ are the **eigenvalues** of the model. To summarize:

> The **eigenvalues** of a model are **scale factors**, normally represented by symbols $\lambda_1, \lambda_2, \lambda_3, \ldots$.

Data Conversion Example – Continued

Problem (2) can be represented as $\mathbf{y} = \boldsymbol{A}\mathbf{x}$, where the diagonal matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{\lambda}_1 & 0 & 0 \\ 0 & \boldsymbol{\lambda}_2 & 0 \\ 0 & 0 & \boldsymbol{\lambda}_3 \end{pmatrix}, \quad \boldsymbol{\lambda}_1 = 1, \ \boldsymbol{\lambda}_2 = \frac{1}{1000}, \ \boldsymbol{\lambda}_3 = \frac{1}{100}.$$

Fourier's model for this matrix $\boldsymbol{A}$ is

$$\boldsymbol{A}\left(\boldsymbol{c}_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \boldsymbol{c}_2\begin{pmatrix}0\\1\\0\end{pmatrix} + \boldsymbol{c}_3\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \boldsymbol{c}_1\boldsymbol{\lambda}_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \boldsymbol{c}_2\boldsymbol{\lambda}_2\begin{pmatrix}0\\1\\0\end{pmatrix} + \boldsymbol{c}_3\boldsymbol{\lambda}_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

History of Fourier's Model

The subject of **eigenanalysis** was popularized by J. B. Fourier in his 1822 publication on the theory of heat, *Théorie analytique de la chaleur*. His ideas can be summarized as follows for the $n \times n$ matrix equation $\mathbf{y} = A\mathbf{x}$.

> The vector $\mathbf{y} = A\mathbf{x}$ is obtained from eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ by replacing the eigenvectors by their scaled versions $\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2, \ldots, \lambda_n\mathbf{v}_n$:
>
> $$\begin{aligned} \mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \quad \text{implies} \\ \mathbf{y} &= x_1\lambda_1\mathbf{v}_1 + x_2\lambda_2\mathbf{v}_2 + \cdots + c_n\lambda_n\mathbf{v}_n. \end{aligned}$$

Determining Equations

The eigenvalues and eigenvectors are determined by homogeneous matrix–vector equations. In Fourier's model

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$$

choose $c_1 = 1$, $c_2 = c_3 = 0$. The equation reduces to $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$. Similarly, taking $c_1 = c_2 = 0$, $c_2 = 1$ implies $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$. Finally, taking $c_1 = c_2 = 0$, $c_3 = 1$ implies $A\mathbf{v}_3 = \lambda_3\mathbf{v}_3$. This proves:

Theorem 1 (Determining Equations in Fourier's Model)
Assume Fourier's model holds. Then the eigenvalues and eigenvectors are determined by the three equations

$$\begin{aligned} A\mathbf{v}_1 &= \lambda_1\mathbf{v}_1, \\ A\mathbf{v}_2 &= \lambda_2\mathbf{v}_2, \\ A\mathbf{v}_3 &= \lambda_3\mathbf{v}_3. \end{aligned}$$

Determining Equations – Continued

The three relations of the theorem can be distilled into one homogeneous matrix–vector equation

$$\boldsymbol{A}\mathbf{v} = \boldsymbol{\lambda}\mathbf{v}.$$

Write it as $\boldsymbol{A}\mathbf{x} - \boldsymbol{\lambda}\mathbf{x} = \mathbf{0}$, then replace $\boldsymbol{\lambda}\mathbf{x}$ by $\boldsymbol{\lambda I}\mathbf{x}$ to obtain the standard form[a]

$$(\boldsymbol{A} - \boldsymbol{\lambda I})\mathbf{v} = \mathbf{0}, \quad \mathbf{v} \neq \mathbf{0}.$$

Let $\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{\lambda I}$. The equation $\boldsymbol{B}\mathbf{v} = \mathbf{0}$ has a nonzero solution **v** if and only if there are infinitely many solutions. Because the matrix is square, infinitely many solutions occurs if and only if $\mathbf{rref}(\boldsymbol{B})$ has a row of zeros. Determinant theory gives a more concise statement: $\det(\boldsymbol{B}) = \mathbf{0}$ if and only if $\boldsymbol{B}\mathbf{v} = \mathbf{0}$ has infinitely many solutions. This proves:

Theorem 2 (Characteristic Equation)

If Fourier's model holds, then the eigenvalues $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, $\boldsymbol{\lambda}_3$ are roots $\boldsymbol{\lambda}$ of the polynomial equation

$$\det(\boldsymbol{A} - \boldsymbol{\lambda I}) = \mathbf{0}.$$

The equation is called the **characteristic equation**. The **characteristic polynomial** is the polynomial on the left, normally obtained by cofactor expansion or the triangular rule.

[a] Identity $\boldsymbol{I}$ is required to factor out the matrix $\boldsymbol{A} - \boldsymbol{\lambda I}$. It is wrong to factor out $\boldsymbol{A} - \boldsymbol{\lambda}$, because $\boldsymbol{A}$ is $\mathbf{3} \times \mathbf{3}$ and $\boldsymbol{\lambda}$ is $\mathbf{1} \times \mathbf{1}$, incompatible sizes for matrix addition.

Theorem 3 (Finding Eigenvectors of A)
For each root $\boldsymbol{\lambda}$ of the characteristic equation, write the frame sequence for $\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{\lambda I}$ with last frame $\mathbf{rref}(\boldsymbol{B})$, followed by solving for the general solution $\mathbf{v}$ of the homogeneous equation $\boldsymbol{B}\mathbf{v} = \mathbf{0}$. Solution $\mathbf{v}$ uses invented parameter names $\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots$. The vector basis answers $\boldsymbol{\partial_{t_1}}\mathbf{v}, \boldsymbol{\partial_{t_2}}\mathbf{v}, \ldots$ are independent **eigenvectors** of $\boldsymbol{A}$ paired to eigenvalue $\boldsymbol{\lambda}$.

Proof: The equation $\boldsymbol{A}\mathbf{v} = \boldsymbol{\lambda}\mathbf{v}$ is equivalent to $\boldsymbol{B}\mathbf{v} = \mathbf{0}$. Because $\det(\boldsymbol{B}) = 0$, then this system has infinitely many solutions, which implies the frame sequence starting at $\boldsymbol{B}$ ends with $\mathbf{rref}(\boldsymbol{B})$ having at least one row of zeros. The general solution then has one or more free variables which are assigned invented symbols $\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots$, and then the vector basis is obtained by from the corresponding list of partial derivatives. Each basis element is a nonzero solution of $\boldsymbol{A}\mathbf{v} = \boldsymbol{\lambda}\mathbf{v}$. By construction, the basis elements (eigenvectors for $\boldsymbol{\lambda}$) are collectively independent. The proof is complete.

Definition 1 (Eigenpair)
An **eigenpair** is an eigenvalue $\boldsymbol{\lambda}$ together with a matching eigenvector $\mathbf{v} \neq \mathbf{0}$ satisfying the equation $\boldsymbol{A}\mathbf{v} = \boldsymbol{\lambda}\mathbf{v}$. The pairing implies that scale factor $\boldsymbol{\lambda}$ is applied to direction $\mathbf{v}$.

A $\mathbf{3} \times \mathbf{3}$ matrix $\boldsymbol{A}$ for which Fourier's model holds has eigenvalues $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, $\boldsymbol{\lambda}_3$ and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. The **eigenpairs** of $\boldsymbol{A}$ are

$$(\boldsymbol{\lambda}_1, \mathbf{v}_1), (\boldsymbol{\lambda}_2, \mathbf{v}_2), (\boldsymbol{\lambda}_3, \mathbf{v}_3).$$

Theorem 4 (Independence of Eigenvectors)
If $(\boldsymbol{\lambda}_1, \mathbf{v}_1)$ and $(\boldsymbol{\lambda}_2, \mathbf{v}_2)$ are two eigenpairs of $\boldsymbol{A}$ and $\boldsymbol{\lambda}_1 \neq \boldsymbol{\lambda}_2$, then $\mathbf{v}_1$, $\mathbf{v}_2$ are independent.
More generally, if $(\boldsymbol{\lambda}_1, \mathbf{v}_1)$, ..., $(\boldsymbol{\lambda}_k, \mathbf{v}_k)$ are eigenpairs of $\boldsymbol{A}$ corresponding to distinct eigenvalues $\boldsymbol{\lambda}_!$, ..., $\boldsymbol{\lambda}_k$, then $\mathbf{v}_1$, ..., $\mathbf{v}_k$ are independent.

The Matrix Eigenanalysis Method

The preceding discussion of data conversion now gives way to abstract definitions which distill the essential theory of eigenanalysis. All of this is algebra, devoid of motivation or application.

Definition 2 (Eigenpair)

A pair $(\lambda, \mathbf{v})$, where $\mathbf{v} \neq \mathbf{0}$ is a vector and λ is a complex number, is called an **eigenpair** of the $n \times n$ matrix A provided

$$A\mathbf{v} = \lambda\mathbf{v} \quad (\mathbf{v} \neq \mathbf{0} \text{ required}). \tag{3}$$

The **nonzero** requirement in (3) results from seeking directions for a coordinate system: the zero vector is not a direction. Any vector $\mathbf{v} \neq \mathbf{0}$ that satisfies (3) is called an **eigenvector** for λ and the value λ is called an **eigenvalue** of the square matrix A.

Eigenanalysis Algorithm

Theorem 5 (Algebraic Eigenanalysis)
Eigenpairs $(\lambda, \mathbf{v})$ of an $n \times n$ matrix A are found by this two-step algorithm:

Step 1 (College Algebra). Solve for eigenvalues λ in the nth order polynomial equation $\det(A - \lambda I) = 0$.

Step 2 (Linear Algebra). For a given root λ from **Step 1**, a corresponding eigenvector $\mathbf{v} \neq \mathbf{0}$ is found by applying the frame sequence method[a] to the homogeneous linear equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

The reported answer for $\mathbf{v}$ is routinely the list of partial derivatives $\partial_{t_1}\mathbf{v}$, $\partial_{t_2}\mathbf{v}$, ..., where t_1, t_2, ... are invented symbols assigned to the free variables.

The reader is asked to apply the algorithm to the identity matrix I; then **Step 1** gives n duplicate answers $\lambda = 1$ and **Step 2** gives n answers, the columns of the identity matrix I.

[a] For $B\mathbf{v} = \mathbf{0}$, the frame sequence begins with B, instead of $\mathbf{aug}(B, \mathbf{0})$. The sequence ends with $\mathbf{rref}(B)$. Then the reduced echelon system is written, followed by assignment of free variables and display of the general solution $\mathbf{v}$.

Proof of the Algebraic Eigneanalysis Theorem

The equation $\boldsymbol{A}\mathbf{v} = \boldsymbol{\lambda}\mathbf{v}$ is equivalent to $(\boldsymbol{A} - \boldsymbol{\lambda}\boldsymbol{I})\mathbf{v} = \mathbf{0}$, which is a set of homogeneous equations, consistent always because of the solution $\mathbf{v} = \mathbf{0}$.

Fix $\boldsymbol{\lambda}$ and define $\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{\lambda}\boldsymbol{I}$. We show that an eigenpair $(\boldsymbol{\lambda}, \mathbf{v})$ exists with $\mathbf{v} \neq \mathbf{0}$ if and only if $\det(\boldsymbol{B}) = 0$, i.e., $\det(\boldsymbol{A} - \boldsymbol{\lambda}\boldsymbol{I}) = 0$. There is a unique solution $\mathbf{v}$ to the homogeneous equation $\boldsymbol{B}\mathbf{v} = \mathbf{0}$ exactly when Cramer's rule applies, in which case $\mathbf{v} = \mathbf{0}$ is the unique solution. All that Cramer's rule requires is $\det(\boldsymbol{B}) \neq 0$. Therefore, an eigenpair exists exactly when Cramer's rule fails to apply, which is when the determinant of coefficients is zero: $\det(\boldsymbol{B}) = 0$.

Eigenvectors for $\boldsymbol{\lambda}$ are found from the general solution to the system of equations $\boldsymbol{B}\mathbf{v} = \mathbf{0}$ where $\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{\lambda}\boldsymbol{I}$. The $\mathbf{rref}$ method produces systematically a reduced echelon system from which the general solution $\mathbf{v}$ is written, depending on invented symbols $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_k$. Since there is never a unique solution, at least one free variable exists. In particular, the last frame $\mathbf{rref}(\boldsymbol{B})$ of the sequence has a row of zeros, which is a useful sanity test.

The **basis of eigenvectors** for $\boldsymbol{\lambda}$ is obtained from the general solution $\mathbf{v}$, which is a linear combination involving the parameters $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_k$. The **basis elements** are reported as the list of partial derivatives $\boldsymbol{\partial}_{t_1}\mathbf{v}, \ldots, \boldsymbol{\partial}_{t_k}\mathbf{v}$.

1 Example (Computing 3×3 Eigenpairs)

Find all eigenpairs of the 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

College Algebra

The eigenvalues are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$, $\lambda_3 = 3$. Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1-\lambda & 2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\ &= ((1-\lambda)^2 + 4)(3-\lambda) && \text{Cofactor rule and Sarrus' rule.} \end{aligned}$$

Root $\lambda = 3$ is found from the factored form above. The roots $\lambda = 1 \pm 2i$ are found from the quadratic formula after expanding $(1-\lambda)^2 + 4 = 0$. Alternatively, take roots across $(\lambda - 1)^2 = -4$.

Linear Algebra

The eigenpairs are

$$\left(1+2i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right), \ \left(1-2i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \ \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Details appear below.

Eigenvector v_1 for $\lambda_1 = 1 + 2i$

$$
\begin{aligned}
B &= A - \lambda_1 I \\
&= \begin{pmatrix} 1-\lambda_1 & 2 & 0 \\ -2 & 1-\lambda_1 & 0 \\ 0 & 0 & 3-\lambda_1 \end{pmatrix} \\
&= \begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \\ 0 & 0 & 2-2i \end{pmatrix} \\
&\approx \begin{pmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Multiply rule.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Combination, factor=}-i\text{.} \\
&\approx \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \text{Swap rule.} \\
&= \mathrm{rref}(A - \lambda_1 I) && \text{Reduced echelon form.}
\end{aligned}
$$

The partial derivative $\partial_{t_1}\mathrm{v}$ of the general solution $x = -it_1$, $y = t_1$, $z = 0$ is eigenvector

$$
\mathrm{v}_1 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.
$$

Eigenvector $\mathbf{v}_2$ for $\boldsymbol{\lambda}_2 = 1 - 2i$

The problem $(\boldsymbol{A} - \boldsymbol{\lambda}_2 \boldsymbol{I})\mathbf{v}_2 = \mathbf{0}$ has solution $\mathbf{v}_2 = \overline{\mathbf{v}_1}$.

To see why, take conjugates across the equation to give $(\overline{\boldsymbol{A}} - \overline{\boldsymbol{\lambda}_2}\boldsymbol{I})\overline{\mathbf{v}_2} = \mathbf{0}$. Then $\overline{\boldsymbol{A}} = \boldsymbol{A}$ ($\boldsymbol{A}$ is real) and $\boldsymbol{\lambda}_1 = \overline{\boldsymbol{\lambda}_2}$ gives $(\boldsymbol{A} - \boldsymbol{\lambda}_1 \boldsymbol{I})\overline{\mathbf{v}_2} = \mathbf{0}$. Then $\overline{\mathbf{v}_2} = \mathbf{v}_1$.

Finally,

$$\mathbf{v}_2 = \overline{\overline{\mathbf{v}}}_2 = \overline{\mathbf{v}_1} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}.$$

Eigenvector v_3 for $\lambda_3 = 3$

$$A - \lambda_3 I = \begin{pmatrix} 1-\lambda_3 & 2 & 0 \\ -2 & 1-\lambda_3 & 0 \\ 0 & 0 & 3-\lambda_3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ Multiply rule.

$$\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ Combination and multiply.

$$= \mathrm{rref}(A - \lambda_3 I)$$ Reduced echelon form.

The partial derivative $\partial_{t_1}\mathbf{v}$ of the general solution $x = 0$, $y = 0$, $z = t_1$ is eigenvector

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$