• Definition: Reduced Echelon System
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Definition 1 (Reduced Echelon System)
A linear system in which each nonzero equation has a lead variable is called a reduced echelon system.

Definition 2 (Rank and Nullity)
The number of lead variables in a reduced echelon system is called the rank of the system. The number of free variables in a reduced echelon system is called the nullity of the system.

We determine the rank and nullity of a system as follows. First, display a frame sequence which starts with that system and ends in a reduced echelon system. Then the rank and nullity of the system are those determined by the final frame.
Theorem 1 (Rank and Nullity)
The following equation holds:

$$\text{rank} + \text{nullity} = \text{number of variables}. $$
Elimination

The elimination algorithm applies at each algebraic step one of the three toolkit rules swap, multiply and combination.

- The objective of each algebraic step is to increase the number of lead variables. The process stops when a signal equation (typically $0 = 1$) is found. Otherwise, it stops when no more lead variables can be found, and then the last system of equations is a reduced echelon system. A detailed explanation of the process has been given in the discussion of frame sequences.

- Reversibility of the algebraic steps means that no solutions are created nor destroyed throughout the algebraic steps: the original system and all systems in the intermediate steps have exactly the same solutions.

- The final reduced echelon system has either a unique solution or infinitely many solutions. In both cases we report the general solution. In the infinitely many solution case, the last frame algorithm is used to write out a general solution.
Theorem 2 (Elimination)
Every linear system has either no solution or else it has exactly the same solutions as an equivalent reduced echelon system, obtained by repeated application of the toolkit rules `swap`, `multiply` and `combination`. 
An Elimination Algorithm

An equation is said to be **processed** if it has a lead variable. Otherwise, the equation is said to be **unprocessed**.

1. If an equation “0 = 0” appears, then move it to the end. If a signal equation “0 = \(c\)” appears (\(c \neq 0\) required), then the system is inconsistent. In this case, the algorithm halts and we report no solution.

2. Identify the **first symbol** \(x_r\), in variable list order \(x_1, \ldots, x_n\), which appears in some unprocessed equation. Apply the **multiply** rule to insure \(x_r\) has leading coefficient one. Apply the **combination** rule to eliminate variable \(x_r\) from all other equations. Then \(x_r\) is a **lead variable**: the number of lead variables has been increased by one.

3. Apply the **swap** rule repeatedly to move this equation past all processed equations, but before the unprocessed equations. Mark the equation as **processed**, e.g., replace \(x_r\) by boxed symbol \([x_r]\).

4. Repeat steps 1–3, until all equations have been processed once. Then lead variables \(x_{i_1}, \ldots, x_{i_m}\) have been defined and the last system is a reduced echelon system.
1 Example (Elimination) Solve the system.

\[
\begin{align*}
    w + 2x - y + z &= 1, \\
    w + 3x - y + 2z &= 0, \\
    x + z &= -1.
\end{align*}
\]

Solution

The answer using the natural variable list order \(w, x, y, z\) is the standard general solution

\[
\begin{align*}
    w &= 3 + t_1 + t_2, \\
    x &= -1 - t_2, \\
    y &= t_1, \\
    z &= t_2, \\
    -\infty &< t_1, t_2 < \infty.
\end{align*}
\]
Details. Elimination will be applied to obtain a frame sequence whose last frame justifies the reported solution. The details amount to applying the three rules swap, multiply and combination for equivalent equations to obtain a last frame which is a reduced echelon system. The standard general solution for the last frame matches the one reported above. Let’s mark processed equations with a box enclosing the lead variable ($w$ is marked $\textcolor{red}{w}$).

1. $w + 2x - y + z = 1$
   $w + 3x - y + 2z = 0$
   $x + z = -1$

2. $w + 2x - y + z = 1$
   $0 + x + 0 + z = -1$
   $x + z = -1$

3. $w + 2x - y + z = 1$
   $x + z = -1$
   $0 = 0$

4. $w + 0 - y - z = 3$
   $x + z = -1$
   $0 = 0$
1. Original system. Identify the variable order as $w, x, y, z$.

2. Choose $w$ as a lead variable. Eliminate $w$ from equation 2 by using $\text{combo}(1, 2, -1)$.

3. The $w$-equation is processed. Let $x$ be the next lead variable. Eliminate $x$ from equation 3 using $\text{combo}(2, 3, -1)$.

4. Eliminate $x$ from equation 1 using $\text{combo}(2, 1, -2)$. Mark the $x$-equation as processed. Reduced echelon system found.

The four frames make the frame sequence which takes the original system into a reduced echelon system. Basic exposition rules apply:

1. Variables in an equation appear in variable list order.
2. Equations inherit variable list order from the lead variables.
The last frame of the sequence, which must be a reduced echelon system, is used to write out the general solution, as follows.

\[
\begin{align*}
  \omega &= 3 + y + z \\
  \omega &= -1 - z \\
  y &= t_1 \\
  z &= t_2 \\
  w &= 3 + t_1 + t_2 \\
  x &= -1 - t_2 \\
  y &= t_1 \\
  z &= t_2
\end{align*}
\]

Solve for the lead variables \( \omega, \omega \). Assign invented symbols \( t_1, t_2 \) to the free variables \( y, z \).

Back-substitute free variables into the lead variable equations to get a standard general solution.