Solution Set Basis for Linear Differential Equations

- Nth Order Linear Differential Equation
- Atoms
- Examples of Atoms
- Theorems about Atoms
  - Atoms are independent
  - Euler's Theorem
  - Basis of the solution set
- How to use Euler's Theorem
- Examples

Linear Differential Equations

The solution set of a homogeneous constant coefficient linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$$

is known to be a vector space of functions of dimension n, consisting of special linear combinations

(1) $$y = c_1 f_1 + \cdots + c_n f_n,$$

where $f_1, \ldots, f_n$ are elementary functions known as **atoms**.

Definition of Atom

An **atom** is defined as the nonzero real or imaginary part of

$$x^n e^{ax}(\cos bx + i \sin bx)$$

with a, b real, $b \geq 0$, and $n = 0, 1, 2 \ldots$ an integer. An atom has coefficient 1, and the zero function is not an atom.

Examples of Atoms

1, x, x^2, e^x, xe^{-x}, $x^{15}e^{2x}\cos 3x$, $\cos 3x$, $\sin 2x$, $x^2\cos 2x$, $x^6\sin \pi x$, $x^{10}e^{\pi x}\sin 0.1x$

Functions that are not Atoms

$x/(1+x)$, $\ln|x|$, e^{x^2}, $\sin(x+1)$, 0, $2x$, $\sin(1/x)$, $\sqrt{x}$

Theorems about Atoms

Theorem 1 (Independence)
Any finite list of atoms is linearly independent.

Theorem 2 (Euler)
The *real* characteristic polynomial

$$p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_0$$

has a factor $(r - a - ib)^{k+1}$ if and only if

$$y = x^k e^{ax}(\cos bx + i \sin bx)$$

is a solution of (1).

Theorem 3 (Basis)
The solution set of equation (1) has a basis of n solution atoms which are determined by Euler's theorem.

Theorem 4 (How to Apply Euler's Theorem)

Factor dividing $p(r)$	Solution Atom(s)
$(r-5)$	e^{5x}
$(r+7)^2$	e^{-7x} and xe^{-7x}
r	e^{0x}
r^2	e^{0x} and xe^{0x}
r^3	$1, x$ and x^2 $[e^{0x}=1]$
$(r-5i)$	$\cos 5x$ and $\sin 5x$
$(r+3i)^2$	$x\cos 3x$ and $x\sin 3x$
$(r-2+3i)^2$	$xe^{2x}\cos 3x$ and $xe^{2x}\sin 3x$

Example 1. Solve $y''' = 0$.

Solution: $p(r) = r^3$ implies 1, x, x^2 are solution atoms. They are independent, hence form a basis for the 3-dimensional solution space.

Example 2. Solve $y'' + 4y = 0$.

Solution: $p(r) = r^2 + 4$ implies $\cos 2x$ and $\sin 2x$ are solution atoms. They are a basis for the 2-dimensional solution space.

Example 3. Solve $y'' + 2y' = 0$.

Solution: $p(r) = r^2 + 2r$ implies 1, e^{-2x} are solution atoms. These independent atoms form a basis for the 2-dimensional solution space.

Example 4. Solve $y^{(4)} + 4y'' = 0$.

Solution: $p(r) = r^4 + 4r^2 = r^2(r^2 + 4)$ implies 1, x, $\cos 2x$, $\sin 2x$ are solution atoms which form a basis for the 4-dimensional solution space.