Basic Theory of Linear Differential Equations

- Picard-Lindelöf Existence-Uniqueness
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- Recipe for Constant-Coefficient Linear Homogeneous Differential Equations
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Theorem 1 (Picard-Lindelöf Existence-Uniqueness)
Let the n-vector function $\mathbf{f}(x, \mathbf{y})$ be continuous for real x satisfying $|x - x_0| \le a$ and for all vectors $\mathbf{y}$ in R^n satisfying $\|\mathbf{y} - \mathbf{y}_0\| \le b$. Additionally, assume that $\partial\mathbf{f}/\partial\mathbf{y}$ is continuous on this domain. Then the initial value problem

$$\begin{cases} \mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \\ \mathbf{y}(x_0) = \mathbf{y}_0 \end{cases}$$

has a unique solution $\mathbf{y}(x)$ defined on $|x - x_0| \le h$, satisfying $\|\mathbf{y} - \mathbf{y}_0\| \le b$, for some constant h, $0 < h < a$.

The unique solution can be written in terms of the *Picard Iterates*

$$\mathbf{y}_{n+1}(x) = \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(t, \mathbf{y}_n(t))dt, \quad \mathbf{y}_0(x) \equiv \mathbf{y}_0,$$

as the formula

$$\mathbf{y}(x) = \mathbf{y}_n(x) + R_n(x), \quad \lim_{n\to\infty} R_n(x) = 0.$$

The formula means $\mathbf{y}(x)$ can be computed as the iterate $\mathbf{y}_n(x)$ for large n.

Theorem 2 (Second Order Linear Picard-Lindelöf Existence-Uniqueness) Let the coefficients $a(x)$, $b(x)$, $c(x)$, $f(x)$ be continuous on an interval J containing $x = x_0$. Assume $a(x) \neq 0$ on J. Let g_1 and g_2 be real constants. The initial value problem

$$\begin{cases} a(x)y'' + b(x)y' + c(x)y = f(x), \\ y(x_0) \ = \ g_1, \\ y'(x_0) \ = \ g_2 \end{cases}$$

has a unique solution $y(x)$ defined on J.

Theorem 3 (Higher Order Linear Picard-Lindelöf Existence-Uniqueness) Let the coefficients $a_0(x)$, …, $a_n(x)$, $f(x)$ be continuous on an interval J containing $x = x_0$. Assume $a_n(x) \neq 0$ on J. Let g_1, …, g_n be constants. Then the initial value problem

$$\begin{cases} a_n(x)y^{(n)}(x) + \cdots + a_0(x)y = f(x), \\ y(x_0) \quad = \quad g_1, \\ y'(x_0) \quad = \quad g_2, \\ \qquad \vdots \\ y^{(n-1)}(x_0) \quad = \quad g_n \end{cases}$$

has a unique solution $y(x)$ defined on J.

Theorem 4 (Homogeneous Structure 2nd Order)
The homogeneous equation $a(x)y''+b(x)y'+c(x)y = 0$ has a general solution of the form

$$y_h(x) = c_1y_1(x) + c_2y_2(x),$$

where c_1, c_2 are arbitrary constants and $y_1(x)$, $y_2(x)$ are independent solutions.

Theorem 5 (Homogeneous Structure nth Order)
The homogeneous equation $a_n(x)y^{(n)}+\cdots+a_0(x)y = 0$ has a general solution of the form

$$y_h(x) = c_1y_1(x) + \cdots + c_ny_n(x),$$

where $c_1, \ldots, c_n$ are arbitrary constants and $y_1(x), \ldots, y_n(x)$ are independent solutions.

Theorem 6 (First Order Recipe)
Let a and b be constants, $a \neq 0$. Let r_1 denote the root of $ar + b = 0$ and construct its corresponding atom $e^{r_1 x}$. Multiply the atom by arbitrary constant c_1. Then $y = c_1 e^{r_1 x}$ is the general solution of the first order equation

$$ay' + by = 0.$$

The equation $ar + b = 0$, called the *characteristic equation*, is found by the formal replacements $y' \longrightarrow r$, $y \longrightarrow 1$ in the differential equation $ay' + by = 0$.

Theorem 7 (Second Order Recipe)
Let $a \neq 0$, b and c be real constant. Then the general solution of

$$ay'' + by' + cy = 0$$

is given by the expression $y = c_1 y_1 + c_2 y_2$, where c_1, c_2 are arbitrary constants and y_1, y_2 are two atoms constructed as outlined below from the roots of the *characteristic equation*

$$ar^2 + br + c = 0.$$

The characteristic equation $ar^2 + br + c = 0$ is found by the formal replacements $y'' \longrightarrow r^2$, $y' \longrightarrow r$, $y \longrightarrow 1$ in the differential equation $ay'' + by' + cy = 0$.

Construction of Atoms for Second Order

The atom construction from the roots r_1, r_2 of $ar^2 + br + c = 0$ is based on Euler's theorem below, organized by the sign of the discriminant $D = b^2 - 4ac$.

$D > 0$ (Real distinct roots $r_1 \neq r_2$)	$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}$.
$D = 0$ (Real equal roots $r_1 = r_2$)	$y_1 = e^{r_1 x}, \quad y_2 = x e^{r_1 x}$.
$D < 0$ (Conjugate roots $r_1 = \overline{r}_2 = \alpha + i\beta$)	$y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$.

Theorem 8 (Euler's Theorem)
The atom $y = x^k e^{\alpha x}\cos\beta x$ is a solution of $ay'' + by' + cy = 0$ if and only if $r_1 = \alpha + i\beta$ is a root of the characteristic equation $ar^2 + br + c = 0$ and $(r - r_1)^k$ divides $ar^2 + br + c$.

Valid also for $\sin\beta x$ when $\beta > 0$. Always, $\beta \geq 0$. For second order, only $k = 1, 2$ are possible.

Euler's theorem is valid for any order differential equation: replace the equation by $a_n y^{(n)} + \cdots + a_0 y = 0$ and the characteristic equation by $a_n r^n + \cdots + a_0 = 0$.

Theorem 9 (Recipe for nth Order)
Let $a_n \neq 0, \ldots, a_0$ be real constants. Let $y_1, \ldots, y_n$ be the list of n distinct atoms constructed by Euler's Theorem from the n roots of the characteristic equation

$$a_n r^n + \cdots + a_0 = 0.$$

Then $y_1, \ldots, y_n$ are independent solutions of

$$a_n y^{(n)} + \cdots + a_0 y = 0$$

and all solutions are given by the general solution formula

$$y = c_1 y_1 + \cdots + c_n y_n,$$

where $c_1, \ldots, c_n$ are arbitrary constants.

The characteristic equation is found by the formal replacements $y^{(n)} \longrightarrow r^n, \ldots, y' \longrightarrow r$, $y \longrightarrow 1$ in the differential equation.

Theorem 10 (Superposition)
The homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ has the *superposition property*:

> If y_1, y_2 are solutions and c_1, c_2 are constants, then the combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is a solution.
>
> The result implies that *linear combinations of solutions are also solutions*.

The theorem applies as well to an nth order linear homogeneous differential equation with continuous coefficients $a_0(x), \ldots, a_n(x)$.

The result can be extended to more than two solutions. If $y_1, \ldots, y_k$ are solutions of the differential equation, then all linear combinations of these solutions are also solutions.

The solution space of a linear homogeneous nth order linear differential equation is a subspace S of the vector space V of all functions on the common domain J of continuity of the coefficients.

Theorem 11 (Non-Homogeneous Structure 2nd Order)
The non-homogeneous equation $a(x)y'' + b(x)y' + c(x)y = f(x)$ has general solution

$$y(x) = y_h(x) + y_p(x),$$

where

- $y_h(x)$ is the general solution of the homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$, and
- $y_p(x)$ is a particular solution of the nonhomogeneous equation $a(x)y'' + b(x)y' + c(x)y = f(x)$.

The theorem is valid for higher order equations: the general solution of the non-homogeneous equation is $y = y_h + y_p$, where y_h is the general solution of the homogeneous equation and y_p is *any* particular solution of the non-homogeneous equation.

An Example

For equation $y'' - y = 10$, the homogeneous equation $y'' - y = 0$ has general solution $y_h = c_1 e^x + c_2 e^{-x}$. Select $y_p = -10$, an equilibrium solution. Then $y = y_h + y_p = c_1 e^x + c_2 e^{-x} - 10$.

Theorem 12 (Non-Homogeneous Structure nth Order)
The non-homogeneous equation $a_n(x)y^{(n)} + \cdots + a_0(x)y = f(x)$ has general solution

$$y(x) = y_h(x) + y_p(x),$$

where

- $y_h(x)$ is the general solution of the homogeneous equation $a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$, and
- $y_p(x)$ is a particular solution of the nonhomogeneous equation $a_n(x)y^{(n)} + \cdots + a_0(x)y = f(x)$.