## Electrical Circuits

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- Voltage drop formulas of Faraday, Ohm, Coulomb.
- Kirchhoff's laws.
- LRC Circuit equation.
- Electrical-Mechanical Analogy.
- Transient and Steady-state Currents.
- Reactance and Impedance.
- Time lag.
- Electrical Resonance.

## Voltage Drop Formulas

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|---|---|
| **Faraday's Law** | $V_L = L\dfrac{dI}{dt}$ |
| | $L =$ inductance in henries, |
| | $I =$ current in amperes. |
| **Ohm's Law** | $V_R = RI$ |
| | $R =$ resistance in ohms. |
| **Coulomb's Law** | $V_C = \dfrac{Q}{C}$ |
| | $Q =$ charge in coulombs, |
| | $C =$ capacitance in farads. |

## Kirchhoff's Laws

The **charge** $\boldsymbol{Q}$ and **current** $\boldsymbol{I}$ are related by the equation

$$\frac{dQ}{dt} = I.$$

- **Loop Law**: *The algebraic sum of the voltage drops around a closed loop is zero.*
- **Junction Law**: *The algebraic sum of the currents at a node is zero.*

## LRC Circuit Equation in Charge form

The first law of Kirchhoff implies the RLC circuit equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

where inductor $L$, resistor $R$ and capacitor $C$ are in a single loop having electromotive force $E(t)$.

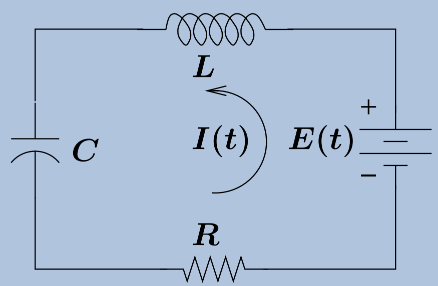


**Figure** 1. An LRC Circuit.

The components are a resistor $R$, inductor $L$, capacitor $C$ and emf $E(t)$. Current $I(t)$ is assigned counterclockwise direction, from minus to plus on the emf terminals.

**LRC Circuit Equation in Current Form**

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Differentiation of the charge form of the LRC circuit equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

gives the current form of the LRC circuit equation

$$LI'' + RI' + \frac{1}{C}I = \frac{dE}{dt}.$$

## Electrical–Mechanical Analogy

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$$\begin{array}{ccccccc} mx'' & + & cx' & + & kx & = & F(t), \\ LQ'' & + & RQ & + & C^{-1}Q & = & E(t). \end{array}$$

**Table 1. Electrical–Mechanical Analogy**

| **Mechanical System** | **Electrical System** |
|---|---|
| Mass $m$ | Inductance $L$ |
| Dampening constant $c$ | Resistance $R$ |
| Hooke's constant $k$ | Reciprocal capacitance $1/C$ |
| Position $x$ | Charge $Q$ [or Current $I$] |
| External force $F$ | Electromotive force $E$ [or $dE/dt$] |

## Transient and Steady-state Currents

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The theory of mechanical systems leads to electrical results by applying the electrical-mechanical analogy to the LRC circuit equation in current form with $E(t) = E_0 \sin \omega t$. We assume $L$, $R$ and $C$ positive.

- The solution $I_h$ of the homogeneous equation $LI'' + RI' + \frac{1}{C} I = 0$ is a **transient current**, satisfying
$$\lim_{t \to \infty} I_h(t) = 0.$$

- The non-homogeneous equation $LI'' + RI' + \frac{1}{C} I = E_0 \omega \cos \omega t$ has a unique periodic solution [**steady-state current**]
$$I_{\mathrm{SS}}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + S^2}}, \quad S \equiv \omega L - \frac{1}{\omega C}, \quad \tan \alpha = \frac{\omega RC}{1 - LC\omega^2}.$$
  It is found by the method of undetermined coefficients.

## Reactance and Impedance

Write

$$I_{\mathrm{SS}}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + S^2}}$$

as

$$I_{\mathrm{SS}}(t) = \frac{E_0}{Z} \cos(\omega t - \alpha)$$

where

$$\begin{aligned} Z &= \sqrt{R^2 + S^2} \quad \text{is called the } \textbf{impedance} \\ S &= \omega L - \frac{1}{\omega C} \quad \text{is called the } \textbf{reactance.} \end{aligned}$$

**Time Lag**

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The steady-state current $\boldsymbol{I_{\mathrm{SS}}(t)\frac{E_0}{Z}\cos(\omega t-\alpha)}$ can be written as a sine function using trigonometric identities:

$$\boldsymbol{I_{\mathrm{SS}}(t)=\frac{E_0}{Z}\sin(\omega t-\delta),\quad \tan\delta=\frac{LC\omega^2-1}{\omega RC}.}$$

Because the input is

$$\boldsymbol{E(t)=E_0\omega\sin(\omega t),}$$

then the **time lag** between the input voltage and the steady-state current is

$$\boldsymbol{\frac{\delta}{\omega}=\frac{1}{\omega}\arctan\left(\frac{LC\omega^2-1}{\omega RC}\right)\quad \text{seconds}.}$$

**Electrical Resonance**

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**Resonance** in an LRC circuit is defined only for sinusoidal inputs $E(t) = E_0 \sin(\omega t)$. Then the differential equation in current form is

$$I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{E_0 \omega}{L} \cos(\omega t).$$

Resonance happens if there is a frequency $\omega$ which maximizes the amplitude $I_0 = E_0/Z$ of the steady-state solution. By calculus, this happens exactly when $dZ/d\omega = 0$, which gives the **resonant frequency**

$$\omega = \frac{1}{\sqrt{LC}}.$$

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**Details**: $dI_0/d\omega = 0$ if and only if $-E_0 Z^{-2} \frac{dZ}{d\omega} = 0$, which is equivalent to $\frac{dZ}{d\omega} = 0$. Then $2S\frac{dS}{d\omega} = 0$ and finally $S = 0$, because $\frac{dS}{d\omega} > 0$. The equation $S = 0$ is equivalent to $\omega = 1/\sqrt{LC}$.