The Corrected Trial Solution
in
the Method of Undetermined Coefficients

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Definition of Related Atoms

Atoms A and B are *related* if and only if their successive derivatives share a common atom.

Then x^3 is related to x and x^{101}, while x is unrelated to e^x, xe^x and $x \sin x$. Atoms $x \sin x$ and $x^3 \cos x$ are related, but the atoms $\cos 2x$ and $\sin x$ are unrelated.

There is an easier way to detect related atoms:

> Atom A is related to atom B if and only if their base atoms are identical or else they would become identical by changing a sine to a cosine.

The Basic Trial Solution Method

The method is outlined here for an nth order linear differential equation.

Undetermined Coefficients Trial Solution Method

Step 1. Let $g(x) = x^n f(x)$, where n is the order of the differential equation. Repeatedly differentiate the atoms of $g(x)$ until no new atoms appear. Collect the distinct atoms so found into a list of k atoms. Multiply these atoms by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** y

.

Step 2. Substitute y into the differential equation.

Step 3. Match coefficients of atoms left and right to write out linear algebraic equations for unknowns $d_1, d_2, \ldots, d_k$. Solve the equations. Any variables not appearing are set to zero.

Step 4. The trial solution y with evaluated coefficients $d_1, d_2, \ldots, d_k$ becomes the particular solution y_p.

Symbols

The symbols $\boldsymbol{c_1}$, $\boldsymbol{c_2}$ are reserved for use as arbitrary constants in the general solution $\boldsymbol{y_h}$ of the homogeneous equation.

Symbols $\boldsymbol{d_1}$, $\boldsymbol{d_2}$, $\boldsymbol{d_3}$, $\ldots$ are reserved for use in the trial solution $\boldsymbol{y}$ of the non-homogeneous equation. Abbreviations: $\boldsymbol{c} =$ constant, $\boldsymbol{d} =$ determined.

Superposition

The relation $y = y_h + y_p$ suggests solving $ay'' + by' + cy = f(x)$ in two stages:

(a) Find y_h as a linear combination of atoms computed by applying Euler's theorem to factors of $ar^2 + br + c$.

(b) Apply the **basic trial solution method** to find y_p.

- We expect to find two arbitrary constants c_1, c_2 in the solution y_h, but in contrast, no arbitrary constants appear in y_p.
- Calling $d_1, d_2, d_3, \ldots$ *undetermined* coefficients is misleading, because in fact they are eventually *determined*.

The Best Trial Solution

Undetermined coefficient theory computes a trial solution with **fewest atoms**, thereby eliminating superfluous symbols, which effects a reduction in the size of the algebra problem. In the case of the example $y'' + y = x^2$, the theory computes a trial solution $y = d_1 + d_2 x + d_3 x^2$, reducing the number of symbols from 5 to 3.

In a general equation $ay'' + by' + cy = f(x)$, the atoms in the trial solution y are the atoms that appear in $g(x) = x^2 f(x)$ plus all lower-power related atoms. Equivalently, the atoms are those extracted from the successive derivatives $g(x)$, $g'(x)$, $g''(x)$, ... For example, if $f(x) = x^2$, then $g(x) = x^2(x^2) = x^4$ and the *list of derivatives* is x^4, $4x^3$, $12x^2$, $24x$, 24. Strip coefficients to identify *the list of related atoms* 1, x, x^2, x^3, x^4.

Two Correction Rules

The *initial* trial solution y obtained by constructing atoms from $g(x) = x^n f(x)$ is not the best trial solution. It is a sum of terms which can be organized into groups of related atoms, and it is known that each group contains n superfluous terms. The correction rules describe how to remove the superfluous terms, which produces the desired corrected trial solution with **fewest possible atoms**.

Correction Rule I

If some variable d_p is missing after substitution **Step 2**, then the system of linear equations for $d_1, \ldots, d_k$ fails to have a unique solution. In the language of linear algebra, a missing variable d_p in the system of linear equations is a *free variable*, which implies the linear system in the unknowns $d_1, \ldots, d_k$ has, among the *three possibilities*, infinitely many solutions.

A symbol d_p appearing in a trial solution will be missing in **Step 2** if and only if it multiplies an atom $A(x)$ that is a solution of the homogeneous equation. Because d_p will be a free variable, to which we will assign value zero in **Step 3**, the term $d_pA(x)$ can be removed from the trial solution. We can do this in advance, to decrease the number of symbols in the trial solution.

Rule I. Remove all terms $d_pA(x)$ in the trial solution of **Step 1** for which atom $A(x)$ is a solution of the homogeneous differential equation.

Correction Rule II

The trial solution always contains superfluous atoms, introduced by using $x^n f(x)$ to construct the trial solution instead of $f(x)$. For example, the equation $y'' + y = x^2$ would have trial solution $y = d_1 + d_2 x + d_3 x^2 + d_4 x^3 + d_5 x^4$, with atoms x^3 and x^4 superfluous, because $y_p = x^2 - 2$. We could have replaced the 5-term trial solution by 3-termed trial solution $y = d_1 + d_2 x + d_3 x^2$. There is a rule for how to remove superfluous terms, which combines easily with Rule I:

Rule II. Terms removed from Rule I appear in groups of related atoms

$$B(x), \quad xB(x), \quad \ldots, \quad x^m B(x),$$

where $B(x)$ is a base atom, that is, an atom not containing a power of x. Rule I removes the first k of these atoms from the trial solution. Rule II removes the last $n - k$ of these atoms. The ones removed are called **superfluous atoms**.

An Illustration

Assume the differential equation has order $n = 2$ and the trial solution contains a sub-list of related atoms

$$e^{2x},\ xe^{2x},\ x^2e^{2x},\ x^3e^{ex}.$$

Example 1

Assume e^{2x} is **not** a solution of the homogeneous equation.

Then Rule I removes no atoms ($k = 0$) and Rule II removes the last 2 atoms ($n - k = 2 - 0 = 2$), resulting in the revised atom sub-list

$$e^{2x},\ xe^{2x}.$$

Example 2

Assume e^{2x} **is** a solution of the homogeneous equation.

Then Rule I removes atom e^{2x} ($k = 1$) from the start of the list and Rule II removes x^3e^{2x} from the end of list ($n - k = 2 - 1 = 1$), resulting in the revised sub-list

$$xe^{2x},\ x^2e^{2x}.$$

Observations

Rule I and Rule II together imply that n atoms are removed from every complete sub-list of related atoms in the original trial solution. The atoms are removed from *the two ends*, killing k from the *beginning* of the list and $n - k$ from the *end* of the list.

As a by-product of the method, the corrected trial solution will have no symbol d_p that ends up as a free variable in the resulting system of linear algebraic equations for the undetermined coefficients. Also, the total number of atoms used in y cannot be reduced.

These facts imply:

> The system of equations has the least possible dimension and a unique solution for the undetermined coefficients.