Variation of Parameters

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Variation of Parameters

The **method of variation of parameters** applies to solve

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{1}$$

- Continuity of a, b, c and f is assumed, plus $a(x) \neq 0$.
- The method is important because it solves the largest class of equations.
- Specifically *included* are functions $f(x)$ like $\ln|x|$, $|x|$, e^{x^2}. Such functions are excluded in the method of undetermined coefficients.

Homogeneous Equation

The method of variation of parameters uses facts about the homogeneous differential equation

(2) $$a(x)y'' + b(x)y' + c(x)y = 0.$$

Success in the method depends upon writing the general solution of (2) as

(3) $$y = c_1 y_1(x) + c_2 y_2(x)$$

where y_1, y_2 are *known functions* and c_1, c_2 are arbitrary constants. If a, b, c are constants, then the standard *recipe* for (2) implies y_1 and y_2 are *independent atoms*.

Independence

Two solutions y_1, y_2 of $a(x)y'' + b(x)y' + c(x)y = 0$ are called **independent** if neither is a constant multiple of the other. The term **dependent** means *not independent*, in which case either $y_1(x) = cy_2(x)$ or $y_2(x) = cy_1(x)$ holds for all x, for some constant c.

Independence can be tested through the **Wronskian** of y_1, y_2, defined by

$$W(x) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Theorem 1 (Wronskian and Independence)
The Wronskian of two solutions satisfies $a(x)W' + b(x)W = 0$, which implies **Abel's identity**

$$W(x) = W(x_0)e^{-\int_{x_0}^{x}(b(t)/a(t))dt}.$$

Two solutions of $a(x)y'' + b(x)y' + c(x)y = 0$ are independent if and only if their Wronskian is nonzero at some point x_0.

Variation of Parameters Formula

Theorem 2 (Variation of Parameters Formula)
Let a, b, c, f be continuous near $x = x_0$ and $a(x) \neq 0$. Let y_1, y_2 be two independent solutions of the homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ and let $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$. Then the non-homogeneous differential equation

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

has a particular solution

$$y_p(x) = \left(\int \frac{y_2(x)(-f(x))}{a(x)W(x)} dx\right) y_1(x) + \left(\int \frac{y_1(x)f(x)}{a(x)W(x)} dx\right) y_2(x).$$

The variation of parameters formula is so named because it expresses $y_p = c_1y_1 + c_2y_2$, where c_1 and c_2 are functions of x, whereas $y_h = c_1y_1 + c_2y_2$ with c_1, c_2 constants.

1 Example (Independence) Consider $y'' - y = 0$. Show the two solutions $\sinh(x)$ and $\cosh(x)$ are independent using Wronskians.

Solution. Let $W(x)$ be the Wronskian of $\sinh(x)$ and $\cosh(x)$. The calculation below shows $W(x) = -1$. By Theorem 1, the solutions are independent.

Background. The calculus *definitions* for hyperbolic functions are

$$\sinh x = (e^x - e^{-x})/2, \quad \cosh x = (e^x + e^{-x})/2.$$

Their derivatives are $(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$. For instance, $(\cosh x)'$ stands for $\frac{1}{2}(e^x + e^{-x})'$, which evaluates to $\frac{1}{2}(e^x - e^{-x})$, or $\sinh x$.

Wronskian detail. Let $y_1 = \sinh x$, $y_2 = \cosh x$. Then

$W = y_1(x)y_2'(x) - y_1'(x)y_2(x)$	Definition of Wronskian W.
$= \sinh(x)\sinh(x) - \cosh(x)\cosh(x)$	Substitute for y_1, y_1', y_2, y_2'.
$= \frac{1}{4}(e^x - e^{-x})^2 - \frac{1}{4}(e^x + e^{-x})^2$	Apply exponential definitions.
$= -1$	Expand and cancel terms.

2 Example (Wronskian) Given $2y'' - xy' + 3y = 0$, verify that a solution pair y_1, y_2 has Wronskian $W(x) = W(0)e^{x^2/4}$.

Solution

Let $a(x) = 2, b(x) = -x, c(x) = 3$. The Wronskian is a solution of

$$W' = -(b/a)W.$$

Then $W' = xW/2$. The solution by growth-decay theory is

$$W = W(0)e^{x^2/4}.$$

3 Example (Variation of Parameters) Solve $y'' + y = \sec x$ by variation of parameters, verifying $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos(x) \ln|\cos x|$.

Solution

Homogeneous solution y_h. Apply the *recipe* for constant equation $y'' + y = 0$. The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$ and $y_h = c_1 \cos x + c_2 \sin x$.

Wronskian. Suitable independent solutions are $y_1 = \cos x$ and $y_2 = \sin x$, taken from the *recipe*. Then $W(x) = \cos^2 x + \sin^2 x = 1$.

Calculate y_p. The variation of parameters formula (2) applies. Integration proceeds near $x = 0$, because $\sec(x)$ is continuous near $x = 0$.

$$\begin{aligned}
y_p(x) &= -y_1(x) \int y_2(x) \sec(x) dx + y_2(x) \int y_1(x) \sec x dx && \boxed{1} \\
&= -\cos x \int \tan(x) dx + \sin x \int 1 dx && \boxed{2} \\
&= x \sin x + \cos(x) \ln|\cos x| && \boxed{3}
\end{aligned}$$

Details: $\boxed{1}$ Use equation (2). $\boxed{2}$ Substitute $y_1 = \cos x$, $y_2 = \sin x$. $\boxed{3}$ Integral tables applied. Integration constants set to zero.