2.1-6 Separate variables and use partial fractions to solve the initial value problem

$$\frac{dx}{dt} = 3x(x-5), \qquad x(0) = 2.$$

$\dfrac{x'}{x(x-5)} = 3$	separated form
$\dfrac{Ax'}{x} + \dfrac{Bx'}{x-5} = 3$	partial fractions applied. constants A, B found later.
$A\ln\lvert x\rvert + B\ln\lvert x-5\rvert = 3t + c$	Method of quadrature
$A\ln 2 + B\ln 3 = c$	Evaluate when $t=0$, $x=2$ to find c
$A\ln\dfrac{\lvert x\rvert}{2} + B\ln\dfrac{\lvert x-5\rvert}{3} = 3t$	Collect terms, using $\ln u - \ln v = \ln(u/v)$
$A = -1/5$, $B = 1/5$ in relation $\dfrac{1}{x(x-5)} = \dfrac{A}{x} + \dfrac{B}{x-5}$	Heaviside's coverup method applied
$-\ln\dfrac{\lvert x\rvert}{2} + \ln\dfrac{\lvert x-5\rvert}{3} = 15t$	Let $A = -1/5$, $B = 1/5$ and multiply by 5
$\ln \dfrac{2}{\lvert x\rvert}\dfrac{\lvert x-5\rvert}{3} = 15t$	use log rule $\ln u - \ln v = \ln(u/v)$
$\dfrac{2}{3}\,\dfrac{5-x}{x} = e^{15t}$	Take logs, drop absolute values near $x=2$
$x = \dfrac{5}{1 + \frac{3}{2}e^{15t}}$	solve for x.

2.1-16

A rabbit population satisfies $P' = aP - bP^2$, $P(0) = P_0$. Given $P_0 = 120$, $aP_0 = 8$ births per month, $bP_0^2 = 6$ deaths per month, Then find The number T of months when $P(T)$ is 95% of The limiting population $M = a/b$.

The given constants are $P_0 = 120$, $a = 1/15$, $b = 6/120^2$, $M = 160$. Then p 80 gives

$$P(T) = \frac{M P_0}{P_0 + (M - P_0) e^{-aT}}$$

and $P(T) = \frac{95}{100} M$ implies The exponential relation

$$\frac{95}{100} M = \frac{P_0 M}{P_0 + (M - P_0) e^{-aT}}$$

Cancel M, clear fractions, rearrange Terms To get

$$e^{-aT} = \frac{5}{95} \frac{P_0}{M - P_0}$$

$$= \frac{5}{95} \cdot \frac{120}{40}$$

$$= \frac{15}{95}$$

Taking logs gives

$$T = 15 \ln\left(\frac{95}{15}\right)$$

2.1-17 $\quad T = \frac{80}{3} \ln\left(\frac{21}{4}\right) \cong 44.22$

#38
P 88

Ref: Edward - Penney section 2.1

Let $P_0 = 76.212$, $P_1 = 123.203$, $P_2 = 179.323$, be the U.S. populations for 1900, 1930, 1960, respectively. Assume the logistic model

$$P' = aP - bP^2,$$

with $M = b/a$, having solution

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-at}} \qquad (1)$$

Fit parameters a, b and then compare $P(80), P(90), P(100)$ to US Census data.

- $e^{-at} = \dfrac{M-P}{M-P_0} \cdot \dfrac{P_0}{P}$
- $e^{-30a} = \dfrac{M-P_1}{M-P_0} \dfrac{P_0}{P_1}$
- $e^{-60a} = \dfrac{M-P_2}{M-P_0} \dfrac{P_0}{P_2}$
- $u = e^{-30a}$
  $u^2 = e^{-60a}$
- $\dfrac{M-P_2}{M-P_0} \dfrac{P_0}{P_2} = \left(\dfrac{M-P_1}{M-P_0} \dfrac{P_0}{P_1}\right)^2$
- $M = 0, 338.03$
  $u = 0.508$
- $a = -\frac{1}{30} \ln u$
  $= 0.0226044519 1$
  $b = Ma$
  $= 7.640914331$

- solve the equation (1) for the exponential term
- substitute $t = 30$
- substitute $t = 60$
- Define u, we'll find u and then $-30a = \ln u$
- substitute to find an equation for M.
- solved in Maple
- used $\ln u = \ln e^{-30a} = -30a$

11-38
cont

$$p(t) = \frac{(338.03)(76.212)}{76.212 + (338.03-76.212)e^{-at}}$$

$$= \frac{25761.7}{76.212 + 261.815\,e^{-at}}$$

$p(80) =$

$p(90) =$

$p(100) =$

p 96
#4
#10

Find The critical points, classify as stable or un sta
and construct The phase diagram. Sketch Typical
curves. Finally, solve The DE explicitly.

2.2-4, p 93. $\frac{dx}{dt} = 3x - x^2$

$f(x) = 3x - x^2$ RHS of The DE.

$0 = x(3-x)$ Find critical points.

$x = 0,\ x = 3$

$f(x)<0$ | $f(x)>0$ | $f(x)<0$ phase diagram (see page 90)

$x=0$ unstable, $x=3$ stable

$$x(t) = \frac{3x_0}{x_0 + (3-x_0)e^{-3t}}$$ See p 91, eqtn (10)

Funnel along $x=3$
Spout along $x=0$ see figure 2.2.4, p 90

2.2-10, p 93. $\frac{dx}{dt} = 7x - x^2 - 10$ (Harvesting)

$f(x) = 7x - x^2 - 10$ RHS of The DE

$0 = 7x - x^2 - 10$

$0 = -(x-5)(x-2)$

$x = 2,\ x = 5$ critical points found

$f(x)<0$ | $f(x)>0$ | $f(x)<0$ phase diagram

$x=2$ unstable, $x=5$ stable

Funnel along $x = 5$ see figure 2.2.8, p 92
Spout along $x = 2$

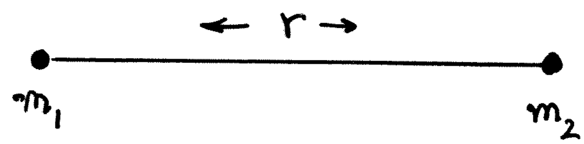
Sketches duplicate The figures on p 90 and p 92.
Submitted solutions should contain copies of These figures.

Escape Velocity of the Earth

Physics Background

Newton's universal law of gravitation

$$\text{Force} = \frac{m_1 m_2 G}{r^2}$$

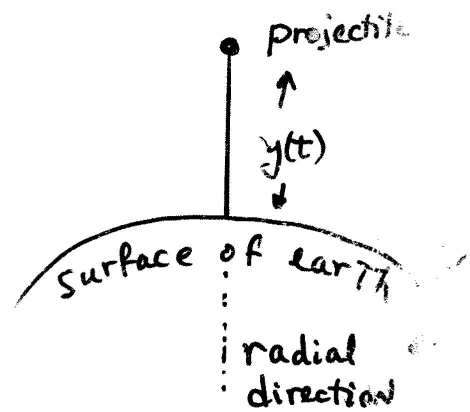


Acceleration g due to gravity

$$g = \frac{G m_2}{R^2}$$

m_2 = Mass of the earth
R = Mean radius of the earth
G = universal Gravitation constant

Differential Equation Theory



- Ignore air resistance
- Let $y(t)$ = projectile-to-surface distance at time t.
- Let m_1 = mass of projectile, m_2 = mass of the earth.

R = mean radius of the earth

DE: $$m_1 y'' = -\frac{G m_1 m_2}{(R+y)^2}$$

IC: $y(0) = 0, \; y'(0) = v_0$

Escape velocity of the Earth

Assumptions

- Velocity v_0 causes the projectile to exit the field of the earth and never return.
- v_0 is minimal.

Math formulation

- $\lim_{t \to \infty} y(t) = \infty$
- v_0 = minimum over all possible v_0

Derivation of $v_0 = \sqrt{2gR}$

$y''y' = -\dfrac{G m_2}{R^2} \dfrac{R^2 y'}{(R+y(t))^2}$	mult DE by y' and cancel m_1
$\frac{1}{2}\left[(y')^2\right]' = -gR^2 \dfrac{y'}{(R+y)^2}$	simplify constants. Write LHS as derivative
$\frac{1}{2}(y')^2 = -gR^2\left(\dfrac{-1}{R+y}\right) + C$	method of quadrature applied.
$\frac{1}{2}v_0^2 = gR + C$	Use $y=0$, $y'=v_0$ @ $t=0$. Found C.
$0 \le C$	Use $(y')^2 \geq 0$ and $\lim_{t\to\infty} y(t) = \infty$.
$0 \le v_0^2 - 2gR$	substitute for C.
$v_0 = \sqrt{2gR}$	Minimum v_0 found.

Ref: Edwards-Penney section 2.3

p 106
#9

Motorboat problem

The equation has the form $v' + pv = q$, which is linear. Solving it gives

$$v = 50(1 - e^{-0.1t}).$$

The limit at ∞ is 50 ft/sec or 34 mph. In your solution:

- solve $1000v' = 5000 - 100v$, $v(0) = 0$.
- Take the limit, watch units.

---

p 107
#27

Projectile problem

The suggestion in the problem can be replaced

Show this step → by "multiply the DE by y' and apply the method of quadrature to variable $v = y'$."

The maximum height is found by setting $y' = 0$ (or $v = 0$). This is the equation

$$0 = R(R+y)v_0^2 - 2GMy,$$

because a fraction $\frac{A}{B}$ is zero only in case $A = 0$.

To solve for y is routine, but to evaluate you must locate values for R, G, M and use the given $v_0 = 1$ km/sec.

$G = 6.6726 \times 10^{-11}$
$M = 5.975 \times 10^{24}$
$R = 6.378 \times 10^{6}$

answer: 51.427 kilometers.