Laplace Table Derivations

- $L(t^n) = \dfrac{n!}{s^{1+n}}$
- $L(e^{at}) = \dfrac{1}{s-a}$
- $L(\cos bt) = \dfrac{s}{s^2+b^2}$
- $L(\sin bt) = \dfrac{b}{s^2+b^2}$
- $L(H(t-a)) = \dfrac{e^{-as}}{s}$
- $L(\delta(t-a)) = e^{-as}$
- $L(\textsf{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$
- $L(\textsf{sqw}(t/a)) = \dfrac{1}{s}\tanh(as/2)$
- $L(a\,\textsf{trw}(t/a)) = \dfrac{1}{s^2}\tanh(as/2)$
- $L(t^\alpha) = \dfrac{\Gamma(1+\alpha)}{s^{1+\alpha}}$
- $L(t^{-1/2}) = \sqrt{\dfrac{\pi}{s}}$

Proof of $L(t^n) = n!/s^{1+n}$
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The first step is to evaluate $L(f(t))$ for $f(t) = t^0$ [$n = 0$ case]. The function t^0 is written as 1, but Laplace theory conventions require $f(t) = 0$ for $t < 0$, therefore $f(t)$ is technically the **unit step function**.

$L(1) = \int_0^\infty (1)e^{-st}dt$	Laplace integral of $f(t) = 1$.
$= -(1/s)e^{-st}\big\vert_{t=0}^{t=\infty}$	Evaluate the integral.
$= 1/s$	Assumed $s > 0$ to evaluate $\lim_{t\to\infty} e^{-st}$.

Proof of $L(t^n) = n!/s^{1+n}$

The value of $L(f(t))$ for $f(t) = t$ can be obtained by s-differentiation of the relation $L(1) = 1/s$, as follows. Technically, $f(t) = 0$ for $t < 0$, then $f(t)$ is called the **ramp function**.

$$
\begin{aligned}
\frac{d}{ds}L(1) &= \frac{d}{ds}\int_0^\infty (1)e^{-st}dt && \text{Laplace integral for } f(t) = 1. \\
&= \int_0^\infty \frac{d}{ds}\left(e^{-st}\right)dt && \text{Used } \frac{d}{ds}\int_a^b F dt = \int_a^b \frac{dF}{ds}dt. \\
&= \int_0^\infty (-t)e^{-st}dt && \text{Calculus rule } (e^u)' = u'e^u. \\
&= -L(t) && \text{Definition of } L(t).
\end{aligned}
$$

Then

$$
\begin{aligned}
L(t) &= -\frac{d}{ds}L(1) && \text{Rewrite last display.} \\
&= -\frac{d}{ds}(1/s) && \text{Use } L(1) = 1/s. \\
&= 1/s^2 && \text{Differentiate.}
\end{aligned}
$$

Proof of $L(t^n) = n!/s^{1+n}$
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This idea can be repeated to give

$$\begin{aligned} L(t^2) &= -\frac{d}{ds}L(t) \\ &= L(t^2) \\ &= \frac{2}{s^3}. \end{aligned}$$

The pattern is $L(t^n) = -\frac{d}{ds}L(t^{n-1})$, which implies the formula

$$L(t^n) = \frac{n!}{s^{1+n}}.$$

The proof is complete.

Proof of $L(e^{at}) = \dfrac{1}{s-a}$

The result follows from $L(1) = 1/s$, as follows.

$L(e^{at}) = \int_0^\infty e^{at}e^{-st}dt$	Direct Laplace transform.
$= \int_0^\infty e^{-(s-a)t}dt$	Use $e^A e^B = e^{A+B}$.
$= \int_0^\infty e^{-St}dt$	Substitute $S = s - a$.
$= 1/S$	Apply $L(1) = 1/s$.
$= 1/(s-a)$	Back-substitute $S = s - a$.

Proof of $L(\cos bt) = \dfrac{s}{s^2+b^2}$ and $L(\sin bt) = \dfrac{b}{s^2+b^2}$

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Use will be made of Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

usually first introduced in trigonometry. In this formula, θ is a real number in radians and $i = \sqrt{-1}$ is the complex unit.

$$e^{ibt}e^{-st} = (\cos bt)e^{-st} + i(\sin bt)e^{-st}$$

Substitute $\theta = bt$ into Euler's formula and multiply by e^{-st}.

$$\int_0^\infty e^{-ibt}e^{-st}dt = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$$

Integrate $t = 0$ to $t = \infty$. Then use properties of integrals.

$$\frac{1}{s-ib} = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$$

Evaluate the left hand side using $L(e^{at}) = 1/(s-a)$, $a = ib$.

Proof of $L(\cos bt) = \dfrac{s}{s^2+b^2}$ and $L(\sin bt) = \dfrac{b}{s^2+b^2}$

$\dfrac{1}{s-ib} = L(\cos bt) + iL(\sin bt)$	Direct Laplace transform definition.
$\dfrac{s+ib}{s^2+b^2} = L(\cos bt) + iL(\sin bt)$	Use complex rule $1/z = \overline{z}/\lvert z\rvert^2$, $z = A+iB$, $\overline{z} = A-iB$, $\lvert z\rvert = \sqrt{A^2+B^2}$.
$\dfrac{s}{s^2+b^2} = L(\cos bt)$	Extract the real part.
$\dfrac{b}{s^2+b^2} = L(\sin bt)$	Extract the imaginary part.

Proof of $L(H(t-a)) = e^{-as}/s$

$L(H(t-a))$	$= \int_0^\infty H(t-a)e^{-st}dt$	Direct Laplace transform. Assume $a \geq 0$.
	$= \int_a^\infty (1)e^{-st}dt$	Because $H(t-a) = 0$ for $0 \leq t < a$.
	$= \int_0^\infty (1)e^{-s(x+a)}dx$	Change variables $t = x + a$.
	$= e^{-as}\int_0^\infty (1)e^{-sx}dx$	Constant e^{-as} moves outside integral.
	$= e^{-as}(1/s)$	Apply $L(1) = 1/s$.

Proof of $L(\delta(t-a)) = e^{-as}$

The *definition* of the delta function is a formal one, in which every occurrence of symbol $\delta(t-a)dt$ under an integrand is replaced by $dH(t-a)$. The differential symbol $dH(t-a)$ is taken in the sense of the Riemann-Stieltjes integral. This integral is defined in Rudin's *Real analysis* for monotonic integrators $\alpha(x)$ as the limit

$$\int_a^b f(x)d\alpha(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)(\alpha(x_n) - \alpha(x_{n-1}))$$

where $x_0 = a$, $x_N = b$ and $x_0 < x_1 < \cdots < x_N$ forms a partition of $[a, b]$ whose mesh approaches zero as $N \to \infty$.

The steps in computing the Laplace integral of the delta function appear below. Admittedly, the proof requires advanced calculus skills and a certain level of mathematical maturity. The reward is a fuller understanding of the Dirac symbol $\delta(x)$.

Proof of $L(\delta(t-a)) = e^{-as}$

$L(\delta(t-a))$	$= \int_0^\infty e^{-st}\delta(t-a)dt$	Laplace integral, $a > 0$ assumed.
	$= \int_0^\infty e^{-st}dH(t-a)$	Replace $\delta(t-a)dt$ by $dH(t-a)$.
	$= \lim_{M\to\infty} \int_0^M e^{-st}dH(t-a)$	Definition of improper integral.
	$= e^{-sa}$	Explained below.

Proof of $L(\delta(t-a)) = e^{-as}$
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To explain the last step, apply the definition of the Riemann-Stieltjes integral:

$$\int_0^M e^{-st} dH(t-a) = \lim_{N\to\infty} \sum_{n=0}^{N-1} e^{-st_n}(H(t_n - a) - H(t_{n-1} - a))$$

where $0 = t_0 < t_1 < \cdots < t_N = M$ is a partition of $[0, M]$ whose mesh $\max_{1\leq n\leq N}(t_n - t_{n-1})$ approaches zero as $N \longrightarrow \infty$. Given a partition, if $t_{n-1} < a \leq t_n$, then $H(t_n - a) - H(t_{n-1} - a) = 1$, otherwise this factor is zero. Therefore, the sum reduces to a single term e^{-st_n}. This term approaches e^{-sa} as $N \longrightarrow \infty$, because t_n must approach a.

Proof of $L(\textsf{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$

The library function **floor** present in computer languages C and Fortran is defined by $\textsf{floor}(x)$ = greatest whole integer $\le x$, e.g., $\textsf{floor}(5.2) = 5$ and $\textsf{floor}(-1.9) = -2$. The computation of the Laplace integral of $\textsf{floor}(t)$ requires ideas from infinite series, as follows.

$$
\begin{aligned}
F(s) &= \int_0^\infty \textsf{floor}(t) e^{-st} dt && \text{Laplace integral definition.} \\
&= \sum_{n=0}^\infty \int_n^{n+1} (n) e^{-st} dt && \text{On } n \le t < n+1,\ \textsf{floor}(t) = n. \\
&= \sum_{n=0}^\infty \frac{n}{s}(e^{-ns} - e^{-ns-s}) && \text{Evaluate each integral.} \\
&= \frac{1-e^{-s}}{s} \sum_{n=0}^\infty n e^{-sn} && \text{Common factor removed.}
\end{aligned}
$$

Proof of $L(\textsf{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$

$$= \frac{x(1-x)}{s} \sum_{n=0}^{\infty} n x^{n-1}$$

Define $x = e^{-s}$.

$$= \frac{x(1-x)}{s} \frac{d}{dx} \sum_{n=0}^{\infty} x^{n}$$

Term-by-term differentiation.

$$= \frac{x(1-x)}{s} \frac{d}{dx} \frac{1}{1-x}$$

Geometric series sum.

$$= \frac{x}{s(1-x)}$$

Compute the derivative, simplify.

$$= \frac{e^{-s}}{s(1-e^{-s})}$$

Substitute $x = e^{-s}$.

Proof of $L(\textbf{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$

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To evaluate the Laplace integral of $\textbf{floor}(t/a)$, a change of variables is made.

$L(\textbf{floor}(t/a))$	$= \int_0^\infty \textbf{floor}(t/a)e^{-st}dt$	Laplace integral definition.
	$= a\int_0^\infty \textbf{floor}(r)e^{-asr}dr$	Change variables $t = ar$.
	$= aF(as)$	Apply the formula for $F(s)$.
	$= \dfrac{e^{-as}}{s(1-e^{-as})}$	Simplify.

Proof of $L(\textbf{sqw}(t/a)) = \dfrac{1}{s}\tanh(as/2)$

The square wave defined by $\textbf{sqw}(x) = (-1)^{\textbf{floor}(x)}$ is periodic of period 2 and piecewise-defined. Let $P = \int_0^2 \textbf{sqw}(t)e^{-st}dt$.

$P = \int_0^1 \textbf{sqw}(t)e^{-st}dt + \int_1^2 \textbf{sqw}(t)e^{-st}dt$	Apply $\int_a^b = \int_a^c + \int_c^b$.
$= \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt$	Use $\textbf{sqw}(x) = 1$ on $0 \le x < 1$ and $\textbf{sqw}(x) = -1$ on $1 \le x < 2$.
$= \dfrac{1}{s}(1 - e^{-s}) + \dfrac{1}{s}(e^{-2s} - e^{-s})$	Evaluate each integral.
$= \dfrac{1}{s}(1 - e^{-s})^2$	Collect terms.

Proof of $L(\mathsf{sqw}(t/a)) = \frac{1}{s}\tanh(as/2)$

Slide 2 of 3 – Compute $L(\mathsf{sqw}(t))$

$L(\mathsf{sqw}(t)) = \dfrac{\int_0^2 \mathsf{sqw}(t)e^{-st}dt}{1-e^{-2s}}$	Periodic function formula.
$= \dfrac{1}{s}(1-e^{-s})^2\dfrac{1}{1-e^{-2s}}.$	Use the computation of P above.
$= \dfrac{1}{s}\dfrac{1-e^{-s}}{1+e^{-s}}.$	Factor $1-e^{-2s} = (1-e^{-s})(1+e^{-s})$.
$= \dfrac{1}{s}\dfrac{e^{s/2}-e^{-s/2}}{e^{s/2}+e^{-s/2}}.$	Multiply the fraction by $e^{s/2}/e^{s/2}$.
$= \dfrac{1}{s}\dfrac{\sinh(s/2)}{\cosh(s/2)}.$	Use $\sinh u = (e^u - e^{-u})/2$, $\cosh u = (e^u + e^{-u})/2$.
$= \dfrac{1}{s}\tanh(s/2).$	Use $\tanh u = \sinh u/\cosh u$.

Proof of $L(\textsf{sqw}(t/a)) = \frac{1}{s}\tanh(as/2)$

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To complete the computation of $L(\textsf{sqw}(t/a))$, a change of variables is made:

$$
\begin{aligned}
L(\textsf{sqw}(t/a)) &= \int_0^\infty \textsf{sqw}(t/a)e^{-st}dt && \text{Direct transform.}\\
&= \int_0^\infty \textsf{sqw}(r)e^{-asr}(a)dr && \text{Change variables } r = t/a.\\
&= \frac{a}{as}\tanh(as/2) && \text{See } L(\textsf{sqw}(t)) \text{ above.}\\
&= \frac{1}{s}\tanh(as/2)
\end{aligned}
$$

Proof of $L(a\,\textbf{trw}(t/a)) = \frac{1}{s^2}\tanh(as/2)$

The triangular wave is defined by $\textbf{trw}(t) = \int_0^t \textbf{sqw}(x)dx$.

$$L(a\,\textbf{trw}(t/a)) = \frac{f(0) + L(f'(t))}{s}$$

Let $f(t) = a\,\textbf{trw}(t/a)$. Use $L(f'(t)) = sL(f(t)) - f(0)$.

$$= \frac{1}{s}L(\textbf{sqw}(t/a))$$

Use $f(0) = 0$, then use $(a\int_0^{t/a}\textbf{sqw}(x)dx)' = \textbf{sqw}(t/a)$.

$$= \frac{1}{s^2}\tanh(as/2)$$

Table entry for **sqw**.

Proof of $L(t^\alpha) = \frac{\Gamma(1+\alpha)}{s^{1+\alpha}}$

$L(t^\alpha) = \int_0^\infty t^\alpha e^{-st} dt$	Definition of Laplace integral.
$= \int_0^\infty (u/s)^\alpha e^{-u} du/s$	Change variables $u = st$, $du = sdt$.
$= \frac{1}{s^{1+\alpha}} \int_0^\infty u^\alpha e^{-u} du$	Because s=constant for u-integration.
$= \frac{1}{s^{1+\alpha}} \Gamma(1+\alpha)$.	Because $\Gamma(x) \equiv \int_0^\infty u^{x-1} e^{-u} du$.

Gamma Function

The *generalized factorial function* $\Gamma(x)$ is defined for $x > 0$ and it agrees with the classical factorial $n! = (1)(2)\cdots(n)$ in case $x = n + 1$ is an integer. In literature, $\alpha!$ means $\Gamma(1+\alpha)$. For more details about the Gamma function, see Abramowitz and Stegun or `maple` documentation.

Proof of $L(t^{-1/2}) = \sqrt{\dfrac{\pi}{s}}$

$$L(t^{-1/2}) = \frac{\Gamma(1 + (-1/2))}{s^{1-1/2}}$$

Apply the previous formula.

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

Use $\Gamma(1/2) = \sqrt{\pi}$.