Laplace Theory Examples

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1 Example (Harmonic oscillator) Solve the initial value problem $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$ by Laplace's method.

Solution:

The solution is $x(t) = \sin t$. The details:

$L(x'') + L(x) = L(0)$	Apply L across the equation.
$sL(x') - x'(0) + L(x) = 0$	Parts rule.
$s[sL(x) - x(0)] - x'(0) + L(x) = 0$	Parts rule again.
$(s^2 + 1)L(x) = 1$	Use $x(0) = 0$, $x'(0) = 1$.
$L(x) = \dfrac{1}{s^2+1}$	Isolate $L(x)$ left.
$= L(\sin t)$	Backward Laplace table.
$x(t) = \sin t$	Lerch's cancelation law.

2 Example (s-Differentiation Rule) Show the steps for $\boldsymbol{L(t^2\, e^{5t}) = \dfrac{2}{(s-5)^3}}$.

Solution:

$\boldsymbol{L(t^2 e^{5t})}$	$\boldsymbol{= \left(-\dfrac{d}{ds}\right)\left(-\dfrac{d}{ds}\right) L(e^{5t})}$	Use s-differentiation.
	$\boldsymbol{= (-1)^2 \dfrac{d}{ds}\dfrac{d}{ds}\left(\dfrac{1}{s-5}\right)}$	Forward Laplace table.
	$\boldsymbol{= \dfrac{d}{ds}\left(\dfrac{-1}{(s-5)^2}\right)}$	Calculus power rule.
	$\boldsymbol{= \dfrac{2}{(s-5)^3}}$	Identity verified.

3 Example (First shifting rule) Show the steps for $L(t^2\, e^{-3t}) = \dfrac{2}{(s+3)^3}$.

Solution:

$$\begin{aligned} L(t^2 e^{-3t}) &= L(t^2)\Big|_{s\to s-(-3)} && \text{First shifting rule.} \\ &= \left(\frac{2}{s^{2+1}}\right)\Bigg|_{s\to s-(-3)} && \text{Forward Laplace table.} \\ &= \frac{2}{(s+3)^3} && \text{Identity verified.} \end{aligned}$$

4 Example (Trigonometric formulas) Show the steps used to obtain these Laplace identities:

(a) $L(t\cos at) = \dfrac{s^2 - a^2}{(s^2+a^2)^2}$

(b) $L(t\sin at) = \dfrac{2sa}{(s^2+a^2)^2}$

(c) $L(t^2\cos at) = \dfrac{2(s^3 - 3sa^2)}{(s^2+a^2)^3}$

(d) $L(t^2\sin at) = \dfrac{6s^2a - a^3}{(s^2+a^2)^3}$

Solution: The details for **(b)** and **(d)** are left as exercises.

The details for **(a)**:

$L(t\cos at) = -(d/ds)L(\cos at)$	Use s-differentiation.
$= -\dfrac{d}{ds}\left(\dfrac{s}{s^2+a^2}\right)$	Forward Laplace table.
$= \dfrac{s^2-a^2}{(s^2+a^2)^2}$	Calculus quotient rule.

The details for **(c)**:

$L(t^2\cos at) = -(d/ds)L((-t)\cos at)$	Use s-differentiation.
$= \dfrac{d}{ds}\left(-\dfrac{s^2-a^2}{(s^2+a^2)^2}\right)$	Result of **(a)**.
$= \dfrac{2s^3-6sa^2)}{(s^2+a^2)^3}$	Calculus quotient rule.

5 Example (Exponentials) Show the steps used to obtain these Laplace identities:

(a) $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$ **(c)** $L(te^{at}\cos bt) = \dfrac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$

(b) $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ **(d)** $L(te^{at}\sin bt) = \dfrac{2b(s-a)}{((s-a)^2+b^2)^2}$

Solution: Left as exercises are **(a)**, **(b)** and **(d)**.
Details for **(c)**:

$$\begin{aligned} L(te^{at}\cos bt) &= L(t\cos bt)\big|_{s\to s-a} && \text{First shifting rule.} \\ &= \left(-\frac{d}{ds}L(\cos bt)\right)\bigg|_{s\to s-a} && \text{Use } s\text{-differentiation.} \\ &= \left(-\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)\right)\bigg|_{s\to s-a} && \text{Forward Laplace table.} \\ &= \left(\frac{s^2-b^2}{(s^2+b^2)^2}\right)\bigg|_{s\to s-a} && \text{Calculus quotient rule.} \\ &= \frac{(s-a)^2-b^2}{((s-a)^2+b^2)^2} && \text{Verified (c).} \end{aligned}$$

6 Example (Hyperbolic functions) Establish these Laplace transform facts about $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$.

(a) $L(\cosh at) = \dfrac{s}{s^2 - a^2}$

(c) $L(t\cosh at) = \dfrac{s^2 + a^2}{(s^2 - a^2)^2}$

(b) $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

(d) $L(t\sinh at) = \dfrac{2as}{(s^2 - a^2)^2}$

Solution: Left as exercises are **(b)** and **(c)**.

The details for **(a)**:

$$L(\cosh at) = \tfrac{1}{2}(L(e^{at}) + L(e^{-at}))$$ Definition and linearity.

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$ Forward Laplace table.

$$= \frac{s}{s^2 - a^2}$$ Identity **(a)** verified.

The details for **(d)**:

$$L(t\sinh at) = -\frac{d}{ds}\left(\frac{a}{s^2 - a^2}\right)$$ Apply the s-differentiation rule.

$$= \frac{a(2s)}{(s^2 - a^2)^2}$$ Calculus power rule; **(d)** verified.

7 Example (s-Differentiation Rule) Solve $L(f(t)) = \dfrac{2s}{(s^2+1)^2}$ for $f(t)$.

Solution:The solution is $f(t) = t\sin t$.

The details:

$$\begin{aligned}
L(f(t)) &= \frac{2s}{(s^2+1)^2} && \\
&= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) && \text{Calculus power rule } (u^n)' = nu^{n-1}u'. \\
&= -\frac{d}{ds}\left(L(\sin t)\right) && \text{Backward Laplace table.} \\
&= L(t\sin t) && \text{Apply the } s\text{-differentiation rule.} \\
f(t) = t\sin t & && \text{Lerch's cancelation law.}
\end{aligned}$$

8 Example (First Shifting Rule I) Solve $L(f(t)) = \dfrac{s+7}{s^2+4s+8}$ for $f(t)$.

Solution:

The answer is $f(t) = e^{-2t}(\cos 2t + \frac{5}{2}\sin 2t)$. The details:

$L(f(t)) = \dfrac{s+7}{(s+2)^2+4}$	Complete the square.
$= \dfrac{S+5}{S^2+4}$	Replace $s+2$ by S.
$= \dfrac{S}{S^2+4} + \dfrac{5}{2}\dfrac{2}{S^2+4}$	Split into table entries.
$= \left.\dfrac{s}{s^2+4} + \dfrac{5}{2}\dfrac{2}{s^2+4}\right\vert_{s \to S = s+2}$	Shifting rule preparation.
$= L\left(\cos 2t + \frac{5}{2}\sin 2t\right)\Big\vert_{s \to S = s+2}$	Forward Laplace table.
$= L(e^{-2t}(\cos 2t + \frac{5}{2}\sin 2t))$	First shifting rule.
$f(t) = e^{-2t}(\cos 2t + \frac{5}{2}\sin 2t)$	Lerch's cancelation law.

9 Example (First Shifting Rule II) Solve $L(f(t)) = \dfrac{s+2}{2^2+2s+2}$ for $f(t)$.

Solution:

The answer is $f(t) = e^{-t}\cos t + e^{-t}\sin t$. The details:

$L(f(t)) = \dfrac{s+2}{s^2+2s+2}$	Signal for this method: the denominator has complex roots.
$= \dfrac{s+2}{(s+1)^2+1}$	Complete the square, denominator.
$= \dfrac{S+1}{S^2+1}$	Substitute S for $s+1$.
$= \dfrac{S}{S^2+1} + \dfrac{1}{S^2+1}$	Split into Laplace table entries.
$= L(\cos t) + L(\sin t)\vert_{s \to S = s+1}$	Forward Laplace table.
$= L(e^{-t}\cos t) + L(e^{-t}\sin t)$	First shift rule.
$f(t) = e^{-t}\cos t + e^{-t}\sin t$	Invoke Lerch's cancelation law.

10 Example (Damped oscillator) Solve by Laplace's method the initial value problem $x'' + 2x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$.

Solution:

The solution is $x(t) = e^{-t}\cos t$. The details:

$L(x'') + 2L(x') + 2L(x) = L(0)$	Apply L across the equation.
$sL(x') - x'(0) + 2L(x') + 2L(x) = 0$	The t-derivative rule on x'.
$s[sL(x) - x(0)] - x'(0)$ $+2[L(x) - x(0)] + 2L(x) = 0$	The t-derivative rule on x.
$(s^2 + 2s + 2)L(x) = 1 + s$	Use $x(0) = 1$, $x'(0) = -1$.
$L(x) = \dfrac{s+1}{s^2 + 2s + 2}$	Divide to isolate $L(x)$.
$= \dfrac{s+1}{(s+1)^2 + 1}$	Complete the square in the denominator.
$= L(\cos t)\big\|_{s \to s+1}$	Forward Laplace table.
$= L(e^{-t}\cos t)$	First shifting rule.
$x(t) = e^{-t}\cos t$	Lerch's cancelation law.

11 Example (Second Shifting Rule I) Show the steps for

$$L(\sin t\, H(t-\pi)) = \frac{-e^{-\pi s}}{s^2+1}.$$

Solution:
The second shifting rule is applied as follows.

$\text{LHS} = L(\sin t\, H(t-\pi))$	Left side of the identity.
$= L(g(t)H(t-a))$	Choose $g(t) = \sin t$, $a = \pi$.
$= e^{-as} L(g(t+a)$	Second form, second shifting theorem.
$= e^{-\pi s} L(\sin(t+\pi))$	Substitute $a = \pi$.
$= e^{-\pi s} L(-\sin t)$	Sum rule $\sin(a+b) = \sin a \cos b + \sin b \cos a$ plus $\sin \pi = 0$, $\cos \pi = -1$.
$= e^{-\pi s} \dfrac{-1}{s^2+1}$	Forward Laplace table.
$= \text{RHS}$	Identity verified.

12 Example (Second Shifting Rule II) Solve $L(f(t)) = e^{-3s}\frac{s+1}{s^2+2s+2}$ for $f(t)$.

Solution:

The answer is $f(t) = e^{3-t}\cos(t-3)H(t-3)$. The details:

$$\begin{aligned}
L(f(t)) &= e^{-3s}\frac{s+1}{(s+1)^2+1} && \text{Complete the square.}\\
&= e^{-3s}\frac{S}{S^2+1} && \text{Replace } s+1 \text{ by } S.\\
&= e^{-3S+3}\left(L(\cos t)\right)\big|_{s\to S=s+1} && \text{Forward Laplace table.}\\
&= e^{3}\left(e^{-3s}L(\cos t)\right)\big|_{s\to S=s+1} && \text{Regroup factor } e^{-3S}.\\
&= e^{3}\left(L(\cos(t-3)H(t-3))\right)\big|_{s\to S=s+1} && \text{Second shifting rule.}\\
&= e^{3}L(e^{-t}\cos(t-3)H(t-3)) && \text{First shifting rule.}\\
f(t) &= e^{3-t}\cos(t-3)H(t-3) && \text{Lerch's cancelation law.}
\end{aligned}$$