

# Systems of Differential Equations and Laplace's Method

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### Solving $x' = Cx$

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Apply  $L$  to each side to obtain  $L(x') = CL(x)$ . Use the parts rule

$$L(x') = sL(x) - x(0)$$

to obtain

$$\begin{aligned} sL(x) - x(0) &= L(Cx) \\ sL(x) - L(Cx) &= x(0) \\ sI L(x) - C L(x) &= x(0) \\ (sI - C)L(x) &= x(0). \end{aligned}$$

### Resolvent

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The inverse of  $sI - C$  is called the **resolvent**, a term invented to describe the equation

$$L(x(t)) = (sI - C)^{-1}x(0).$$

## An Illustration for $\mathbf{x}' = C\mathbf{x}$

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Define  $C = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ ,  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ,  $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ , which gives a scalar initial value problem

$$\begin{cases} x_1'(t) = 2x_1(t) + 3x_2(t), \\ x_2'(t) = 4x_2(t), \\ x_1(0) = 1, \\ x_2(0) = 2. \end{cases}$$

Then the adjugate formula  $A^{-1} = \frac{\mathbf{adj}(A)}{\det(A)}$  gives the resolvent

$$(sI - C)^{-1} = \frac{1}{(s-2)(s-4)} \begin{pmatrix} s-4 & 3 \\ 0 & s-2 \end{pmatrix}.$$

The Laplace transform of the solution is then

$$L(\mathbf{x}(t)) = (sI - C)^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{s + 2}{(s - 2)(s - 4)} \\ \frac{2}{s - 4} \end{pmatrix}.$$

Partial fractions and use of the backward Laplace table imply

$$\mathbf{x}(t) = \begin{pmatrix} 3e^{4t} - 2e^{2t} \\ 2e^{4t} \end{pmatrix}.$$