Forced Undamped Oscillations

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Forced Undamped Motion

The equation for study is a forced spring–mass system

$$mx''(t) + kx(t) = f(t).$$

The model originates by equating the Newton's second law force $mx''(t)$ to the sum of the Hooke's force $-kx(t)$ and the external force $f(t)$. The physical model is a laboratory box containing an undamped spring–mass system, transported on a truck as in Figure 1, with external force $f(t) = F_0 \cos \omega t$ induced by the speed bumps.

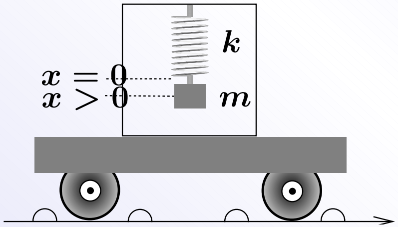

Figure 1. An undamped spring-mass system in a box is transported on a truck. Speed bumps on the shoulder of the road induce periodic vertical oscillations to the box.

Undamped Spring-Mass System

The forced spring-mass equation without damping is

$$x''(t) + \omega_0^2\, x(t) = \frac{F_0}{m}\cos\omega t, \quad \omega_0 = \sqrt{k/m}.$$

The general solution $x(t)$ always presents itself in two pieces, as the sum of the homogeneous solution x_h and a particular solution x_p. For $\omega \neq \omega_0$, the general solution is

$$\begin{aligned} x(t) &= x_h(t) + x_p(t), \\ x_h(t) &= c_1\cos\omega_0 t + c_2\sin\omega_0 t, \quad c_1, c_2 \text{ constants}, \\ x_p(t) &= A_1\cos\omega t, \qquad A_1 = \frac{F_0/m}{\omega_0^2 - \omega^2}. \end{aligned} \tag{1}$$

A general statement can be made about the solution decomposition:

> The solution is a sum of two harmonic oscillations, one of natural frequency ω_0 due to the spring and the other of natural frequency ω due to the external force $F_0\cos\omega t$.

Rapidly and slowly varying functions

The superposition $x(t)$ in (1) will exhibit the phenomenon of **beats** for certain choices of ω_0, ω, $x(0)$ and $x'(0)$. For example, consider $x(t) = \cos\omega_0 t - \cos\omega t$. Use the trigonometric identity $2\sin a \sin b = \cos(a-b) - \cos(a+b)$ to write $x(t) = A(t)\sin\frac{1}{2}(\omega_0+\omega)t$ where $A(t) = 2\sin\frac{1}{2}(\omega_0-\omega)t$. If $\omega \approx \omega_0$, then $A(t)$ has natural frequency $\alpha = \frac{1}{2}(\omega_0-\omega)$ near zero. The natural frequency $\beta = \frac{1}{2}(\omega_0+\omega)$ can be relatively large and therefore $x(t)$ is a product of a **slowly varying** amplitude $A(t) = 2\sin\alpha t$ and a **rapidly varying** oscillation $\sin\beta t$.
The physical phenomenon of **beats** refers to the periodic cancelation of sound at a slow frequency. An illustration of the graphical meaning of *beats* appears in Figure 2.

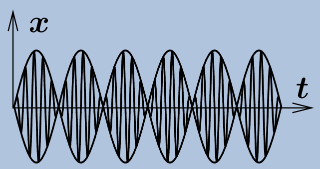

Figure 2. The phenomenon of beats. Shown is a rapidly–varying periodic oscillation $x(t) = 2\sin 4t \sin 40t$ and the two slowly–varying envelope curves $x_1(t) = 2\sin 4t$, $x_2(t) = -2\sin 4t$.

Rotating drum on a cart

Figure 3 shows a model for a rotating machine, like a front–loading clothes dryer. For modeling purposes, the rotating drum with load is replaced by an idealized model: a mass M on a string of radius R rotating with angular speed ω. The center of rotation is located along the center–line of the cart. The total mass m of the cart includes the rotating mass M, which we imagine to be an off–center lump of wet laundry inside the dryer drum.

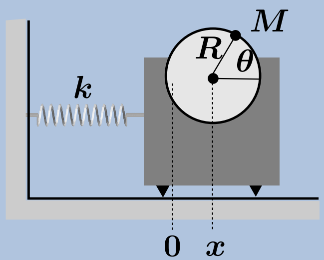

Figure 3. A rotating vertical drum installed on a cart with skids.

Vibrations cause the cart to skid left or right. A spring of Hooke's constant k restores the cart to its equilibrium position $x = 0$. The cart has position $x > 0$ corresponding to skidding distance x to the right of the equilibrium position, due to the off-center load. Similarly, $x < 0$ means the cart skidded distance $|x|$ to the left.

The undamped oscillator model is

$$mx''(t) + kx(t) = RM\omega^2 \cos \omega t. \tag{2}$$

Model Derivation

Friction ignored, Newton's second law gives force $F = m\overline{x}''(t)$, where $\overline{x}$ locates the cart's center of mass. Hooke's law gives force $F = -kx(t)$. The centroid $\overline{x}$ can be expanded in terms of $x(t)$ by using calculus moment of inertia formulas. Let $m_1 = m - M$ be the cart mass, $m_2 = M$ the drum mass, $x_1 = x(t)$ the moment arm for m_1 and $x_2 = x(t) + R\cos\theta$ the moment arm for m_2. Then $\theta = \omega t$ in Figure 3 gives

(3)
$$\begin{aligned}\overline{x}(t) &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \\ &= \frac{(m-M)x(t) + M(x(t) + R\cos\theta)}{m} \\ &= x(t) + \frac{RM}{m}\cos\omega t.\end{aligned}$$

Force competition $m\overline{x}'' = -kx$ and derivative expansion results in the forced harmonic oscillator

$$mx''(t) + kx(t) = RM\omega^2 \cos\omega t.$$