Elementary Matrices and Frame Sequences

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**Elementary Matrices**

**Definition.** An elementary matrix $E$ is the result of applying a combination, multiply or swap rule to the identity matrix.

An elementary matrix is then the *second frame* after a *combo*, *swap* or *mult toolkit* operation which has been applied to a *first frame* equal to the identity matrix.

**Example:**

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]  
First frame = identity matrix.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-5 & 0 & 1 \\
\end{pmatrix}
\]  
Second frame

Elementary combo matrix

\text{combo}(1, 3, -5)
Computer algebra systems and elementary matrices

The computer algebra system Maple displays typical $4 \times 4$ elementary matrices (C=Combination, M=Multiply, S=Swap) as follows.

```
with(linalg):
Id := diag(1,1,1,1);
C := addrow(Id, 2, 3, c);
M := mulrow(Id, 3, m);
S := swaprow(Id, 1, 4);
```

The answers:

\[
C = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & c & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & m & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
S = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]
Constructing elementary matrices $E$ and their inverses $E^{-1}$

- **Mult** Change a one in the identity matrix to symbol $m \neq 0$.
- **Combo** Change a zero in the identity matrix to symbol $c$.
- **Swap** Interchange two rows of the identity matrix.

Constructing $E^{-1}$ from elementary matrix $E$

- **Mult** Change diagonal multiplier $m \neq 0$ in $E$ to $1/m$.
- **Combo** Change multiplier $c$ in $E$ to $-c$.
- **Swap** The inverse of $E$ is $E$ itself.
Fundamental Theorem on Elementary Matrices

Theorem 1 (Frame sequences and elementary matrices)
In a frame sequence, let the second frame $A_2$ be obtained from the first frame $A_1$ by a combo, swap or mult toolkit operation. Let $n$ equal the row dimension of $A_1$. Then there is correspondingly an $n \times n$ combo, swap or mult elementary matrix $E$ such that

$$A_2 = EA_1.$$ 

Theorem 2 (The rref and elementary matrices)
Let $A$ be a given matrix of row dimension $n$. Then there exist $n \times n$ elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$\text{rref}(A) = E_k \cdots E_2E_1 A.$$
Proof of Theorem 1

The first result is the observation that left multiplication of matrix $A_1$ by elementary matrix $E$ gives the answer $A_2 = EA_1$ which is obtained by applying the corresponding combo, swap or mult toolkit operation. This fact is discovered by doing examples, then a formal proof can be constructed (not presented here).

Proof of Theorem 2

The second result applies the first result multiple times to obtain elementary matrices $E_1, E_2, \ldots$ which represent the multiply, combination and swap operations performed in the frame sequence which take the First Frame $A_1 = A$ into the Last Frame $A_{k+1} = \text{rref}(A_1)$. Combining the identities

$$A_2 = E_1A_1, \quad A_3 = E_2A_2, \quad \ldots, \quad A_{k+1} = E_kA_k$$

gives the matrix multiply equation

$$A_{k+1} = E_kE_{k-1}\cdots E_2E_1A_1$$

or equivalently the theorem’s result, because $A_{k+1} = \text{rref}(A)$ and $A_1 = A$. 
A certain 6-frame sequence.

\[
A_1 = \begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 0 \\
3 & 6 & 3
\end{pmatrix} \quad \text{Frame 1, original matrix.}
\]

\[
A_2 = \begin{pmatrix}
1 & 2 & 3 \\
0 & 0 & -6 \\
3 & 6 & 3
\end{pmatrix} \quad \text{Frame 2, combo(1,2,-2).}
\]

\[
A_3 = \begin{pmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
3 & 6 & 3
\end{pmatrix} \quad \text{Frame 3, mult(2,-1/6).}
\]

\[
A_4 = \begin{pmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
0 & 0 & -6
\end{pmatrix} \quad \text{Frame 4, combo(1,3,-3).}
\]

\[
A_5 = \begin{pmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \quad \text{Frame 5, combo(2,3,-6).}
\]

\[
A_6 = \begin{pmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \quad \text{Frame 6, combo(2,1,-3). Found \(\text{rref}(A_1)\).
The corresponding $3 \times 3$ elementary matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 2, combo(1,2,-2) applied to } I.$$  

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 3, mult(2,-1/6) applied to } I.$$  

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad \text{Frame 4, combo(1,3,-3) applied to } I.$$  

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \quad \text{Frame 5, combo(2,3,-6) applied to } I.$$  

$$E_5 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 6, combo(2,1,-3) applied to } I.$$
Frame Sequence Details

\[ A_2 = E_1 A_1 \]
Frame 2, \( E_1 \) equals combo(1,2,-2) on \( I \).

\[ A_3 = E_2 A_2 \]
Frame 3, \( E_2 \) equals mult(2,-1/6) on \( I \).

\[ A_4 = E_3 A_3 \]
Frame 4, \( E_3 \) equals combo(1,3,-3) on \( I \).

\[ A_5 = E_4 A_4 \]
Frame 5, \( E_4 \) equals combo(2,3,-6) on \( I \).

\[ A_6 = E_5 A_5 \]
Frame 6, \( E_5 \) equals combo(2,1,-3) on \( I \).

\[ A_6 = E_5 E_4 E_3 E_2 E_1 A_1 \]
Summary frames 1-6.

Then

\[ \text{rref}(A_1) = E_5 E_4 E_3 E_2 E_1 A_1, \]

which is the result of the Theorem.
Fundamental Theorem Illustrated

The summary:

\[ A_6 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_1 \]

Because \( A_6 = \text{rref}(A_1) \), the above equation gives the inverse relationship

\[ A_1 = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} \text{rref}(A_1). \]

Each inverse matrix is simplified by the rules for constructing \( E^{-1} \) from elementary matrix \( E \), the result being

\[ A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{rref}(A_1) \]
Theorem 3 (RREF Inverse Method)

\[
\text{rref}(\text{aug}(A, I)) = \text{aug}(I, B) \quad \text{if and only if} \quad AB = I.
\]

**Proof:** For *any* matrix $E$ there is the matrix multiply identity

\[
E \text{aug}(C, D) = \text{aug}(EC, ED).
\]

This identity is proved by arguing that each side has identical columns. For example, $\text{col}($LHS, $1) = E \text{col}(C, 1) = \text{col}($RHS, $1)$.

Assume $C = \text{aug}(A, I)$ satisfies $\text{rref}(C) = \text{aug}(I, B)$. The fundamental theorem of elementary matrices implies $E_k \cdots E_1 C = \text{rref}(C)$. Then

\[
\text{rref}(C) = \text{aug}(E_k \cdots E_1 A, E_k \cdots E_1 I) = \text{aug}(I, B)
\]

implies that $E_k \cdots E_1 A = I$ and $E_k \cdots E_1 I = B$. Together, $AB = I$ and then $B$ is the inverse of $A$.

Conversely, assume that $AB = I$. Then $A$ has inverse $B$. The fundamental theorem of elementary matrices implies the identity $E_k \cdots E_1 A = \text{rref}(A) = I$. It follows that $B = E_k \cdots E_1$. Then $\text{rref}(C) = E_k \cdots E_1 \text{aug}(A, I) = \text{aug}(E_k \cdots E_1 A, E_k \cdots E_1 I) = \text{aug}(I, B)$. 
