

## Algebraic Eigenanalysis

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## **The Matrix Eigenanalysis Method**

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The preceding discussion of data conversion now gives way to synthetic abstract definitions which distill the essential theory of eigenanalysis.

All of this is algebra, devoid of motivation or application.

## Eigenpairs

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### Definition 1 (Eigenpair)

A pair  $(\lambda, \mathbf{v})$ , where  $\mathbf{v} \neq \mathbf{0}$  is a vector and  $\lambda$  is a complex number, is called an **eigenpair** of the  $n \times n$  matrix  $A$  provided

$$(1) \quad A\mathbf{v} = \lambda\mathbf{v} \quad (\mathbf{v} \neq \mathbf{0} \text{ required}).$$

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- The **nonzero** requirement in (1) results from seeking directions for a coordinate system: the zero vector is not a direction.
- Any vector  $\mathbf{v} \neq \mathbf{0}$  that satisfies (1) is called an **eigenvector** for  $\lambda$  and the value  $\lambda$  is called an **eigenvalue** of the square matrix  $A$ .

## Eigenanalysis Algorithm

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### Theorem 1 (Algebraic Eigenanalysis)

Eigenpairs  $(\lambda, \mathbf{v})$  of an  $n \times n$  matrix  $A$  are found by this two-step algorithm:

**Step 1 (College Algebra).** Solve for eigenvalues  $\lambda$  in the  $n$ th order polynomial equation  $\det(A - \lambda I) = 0$ .

**Step 2 (Linear Algebra).** For a given root  $\lambda$  from **Step 1**, a corresponding eigenvector  $\mathbf{v} \neq \mathbf{0}$  is found by applying the frame sequence method<sup>a</sup> to the homogeneous linear equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

The answer for  $\mathbf{v}$  is the list of partial derivatives  $\partial_{t_1}\mathbf{v}$ ,  $\partial_{t_2}\mathbf{v}$ ,  $\dots$ , where  $t_1$ ,  $t_2$ ,  $\dots$  are invented symbols assigned to the free variables.

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The reader is asked to apply the algorithm to the identity matrix  $I$ ; then **Step 1** gives  $n$  duplicate answers  $\lambda = 1$  and **Step 2** gives  $n$  answers, the columns of the identity matrix  $I$ .

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<sup>a</sup> For  $B\mathbf{v} = \mathbf{0}$ , the frame sequence begins with  $B$ , instead of  $\text{aug}(B, \mathbf{0})$ . The sequence ends with  $\text{rref}(B)$ . Then the reduced echelon system is written, followed by assignment of free variables and display of the general solution  $\mathbf{v}$ .

## Proof of the Algebraic Eigenanalysis Theorem

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The equation  $A\mathbf{v} = \lambda\mathbf{v}$  is equivalent to  $(A - \lambda I)\mathbf{v} = \mathbf{0}$ , which is a set of homogeneous equations, consistent always because of the solution  $\mathbf{v} = \mathbf{0}$ .

Fix  $\lambda$  and define  $B = A - \lambda I$ . We show that an eigenpair  $(\lambda, \mathbf{v})$  exists with  $\mathbf{v} \neq \mathbf{0}$  if and only if  $\det(B) = 0$ , i.e.,  $\det(A - \lambda I) = 0$ . There is a unique solution  $\mathbf{v}$  to the homogeneous equation  $B\mathbf{v} = \mathbf{0}$  exactly when Cramer's rule applies, in which case  $\mathbf{v} = \mathbf{0}$  is the unique solution. All that Cramer's rule requires is  $\det(B) \neq 0$ . Therefore, an eigenpair exists exactly when Cramer's rule fails to apply, which is when the determinant of coefficients is zero:  $\det(B) = 0$ .

Eigenvectors for  $\lambda$  are found from the general solution to the system of equations  $B\mathbf{v} = \mathbf{0}$  where  $B = A - \lambda I$ . The `rref` method produces systematically a reduced echelon system from which the general solution  $\mathbf{v}$  is written, depending on invented symbols  $t_1, \dots, t_k$ . Since there is never a unique solution, at least one free variable exists. In particular, the last frame `rref(B)` of the sequence has a row of zeros, which is a useful sanity test.

The **basis of eigenvectors** for  $\lambda$  is obtained from the general solution  $\mathbf{v}$ , which is a linear combination involving the parameters  $t_1, \dots, t_k$ . The **basis elements** are reported as the list of partial derivatives  $\partial_{t_1}\mathbf{v}, \dots, \partial_{t_k}\mathbf{v}$ .

## Eigenpair Packages

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The eigenpairs of a  $3 \times 3$  matrix for which Fourier's model holds are labeled

$$(\lambda_1, \mathbf{v}_1), \quad (\lambda_2, \mathbf{v}_2), \quad (\lambda_3, \mathbf{v}_3).$$

An **eigenvector package** is a matrix package  $P$  of eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  given by

$$P = \text{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3).$$

An **eigenvalue package** is a matrix package  $D$  of eigenvalues given by

$$D = \text{diag}(\lambda_1, \lambda_2, \lambda_3).$$

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Important is the *pairing* that is inherited from the eigenpairs, which dictates the packaging order of the eigenvectors and eigenvalues. Matrices  $P, D$  are **not unique**: possible are  $3!$  ( $= 6$ ) column permutations.

## Data Conversion Example

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The eigenvalues for the  $3 \times 3$  data conversion problem are  $\lambda_1 = 1$ ,  $\lambda_2 = 0.001$ ,  $\lambda_3 = 0.01$  and the eigenvectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$  are the columns of the identity matrix  $I$ . Then the eigenpair packages are

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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## Theorem 2 (Eigenpair Packages)

Let  $P$  be a matrix package of eigenvectors and  $D$  the corresponding matrix package of eigenvalues. Then for all vectors  $\mathbf{c}$ ,

$$AP\mathbf{c} = PD\mathbf{c}.$$

## Proof of the Eigenpair Package Theorem

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**Proof:** The result is valid for  $n \times n$  matrices.

We prove the eigenpair package theorem for  $3 \times 3$  matrices. The two sides of the equation are expanded as follows.

$$PD\mathbf{c} = P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{Expand RHS.}$$

$$= P \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \lambda_3 c_3 \end{pmatrix}$$

$$= \lambda_1 c_1 \mathbf{v}_1 + \lambda_2 c_2 \mathbf{v}_2 + \lambda_3 c_3 \mathbf{v}_3$$

Because  $P$  has columns  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .

$$AP\mathbf{c} = A(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3)$$

Expand LHS.

$$= c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2 + c_3 \lambda_3 \mathbf{v}_3$$

Fourier's model.



## The Equation $AP = PD$

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The question of Fourier's model holding for a given  $3 \times 3$  matrix  $A$  is settled here. According to the result, a matrix  $A$  for which Fourier's model holds can be constructed by the formula  $A = PDP^{-1}$  where  $D$  is any diagonal matrix and  $P$  is an invertible matrix.

### Theorem 3 ( $AP = PD$ )

Fourier's model  $A(c_1v_1 + c_2v_2 + c_3v_3) = c_1\lambda_1v_1 + c_2\lambda_2v_2 + c_3\lambda_3v_3$  holds for eigenpairs  $(\lambda_1, v_1)$ ,  $(\lambda_2, v_2)$ ,  $(\lambda_3, v_3)$  if and only if the packages

$$P = \text{aug}(v_1, v_2, v_3), \quad D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$$

satisfy the two requirements

1. Matrix  $P$  is invertible, e.g.,  $\det(P) \neq 0$ .
2. Matrix  $A = PDP^{-1}$ , or equivalently,  $AP = PD$ .

## Proof Details for $AP = PD$

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Assume Fourier's model holds. Define  $P$  and  $D$  to be the eigenpair packages. Then **1** holds, because the columns of  $P$  are independent. By Theorem **2**,  $APc = PDc$  for all vectors  $c$ . Taking  $c$  equal to a column of the identity matrix  $I$  implies the columns of  $AP$  and  $PD$  are identical, that is,  $AP = PD$ . A multiplication of  $AP = PD$  by  $P^{-1}$  gives **2**.

Conversely, let  $P$  and  $D$  be given packages satisfying **1**, **2**. Define  $v_1, v_2, v_3$  to be the columns of  $P$ . Then the columns pass the rank test, because  $P$  is invertible, proving independence of the columns. Define  $\lambda_1, \lambda_2, \lambda_3$  to be the diagonal elements of  $D$ . Using  $AP = PD$ , we calculate the two sides of  $APc = PDc$  as in the proof of Theorem **2**, which shows that  $x = c_1v_1 + c_2v_2 + c_3v_3$  implies  $Ax = c_1\lambda_1v_1 + c_2\lambda_2v_2 + c_3\lambda_3v_3$ . Hence Fourier's model holds.

## Diagonalization

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A square matrix  $\mathbf{A}$  is called **diagonalizable** provided  $\mathbf{AP} = \mathbf{PD}$  for some diagonal matrix  $\mathbf{D}$  and invertible matrix  $\mathbf{P}$ . The preceding discussions imply that  $\mathbf{D}$  must be a package of eigenvalues of  $\mathbf{A}$  and  $\mathbf{P}$  must be the corresponding package of eigenvectors of  $\mathbf{A}$ . The requirement on  $\mathbf{P}$  to be invertible is equivalent to asking that the eigenvectors of  $\mathbf{A}$  be independent and equal in number to the column dimension of  $\mathbf{A}$ .

The matrices  $\mathbf{A}$  for which Fourier's model is valid is precisely the class of diagonalizable matrices. This class is not the set of all square matrices: there are matrices  $\mathbf{A}$  for which Fourier's model is invalid. They are called **non-diagonalizable matrices**.

## Distinct Eigenvalues and Diagonalization

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The construction for eigenvector package  $P$  always produces independent columns. Even if  $A$  has fewer than  $n$  eigenpairs, the construction still produces independent eigenvectors. In such **non-diagonalizable** cases, caused by insufficient columns to form  $P$ , matrix  $A$  must have an eigenvalue of multiplicity greater than one.

If all eigenvalues are distinct, then the correct number of independent eigenvectors were found and  $A$  is then **diagonalizable** with packages  $D, P$  satisfying  $AP = PD$ . This proves the following result.

### **Theorem 4 (Distinct Eigenvalues)**

If an  $n \times n$  matrix  $A$  has  $n$  distinct eigenvalues, then it has  $n$  eigenpairs and  $A$  is diagonalizable with eigenpair packages  $D, P$  satisfying  $AP = PD$ .

## 1 Example (Computing $2 \times 2$ Eigenpairs)

Find all eigenpairs of the  $2 \times 2$  matrix  $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ .

### Solution:

**College Algebra.** The eigenvalues are  $\lambda_1 = 1$ ,  $\lambda_2 = -1$ . Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{vmatrix} 1 - \lambda & 0 \\ 2 & -1 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(-1 - \lambda) \end{aligned}$$

Characteristic equation.

Subtract  $\lambda$  from the diagonal.

Sarrus' rule.

## Solution:

**Linear Algebra.** The eigenpairs are  $\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \left(-1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ . Details:

**Eigenvector for  $\lambda_1 = 1$ .**

$$\begin{aligned} A - \lambda_1 I &= \begin{pmatrix} 1 - \lambda_1 & 0 \\ 2 & -1 - \lambda_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\ &= \text{rref}(A - \lambda_1 I) \end{aligned}$$

Swap and multiply rules.

Reduced echelon form.

The partial derivative  $\partial_{t_1} \mathbf{v}$  of the general solution  $x = t_1, y = t_1$  is eigenvector  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

**Eigenvector for  $\lambda_2 = -1$ .**

$$\begin{aligned} A - \lambda_2 I &= \begin{pmatrix} 1 - \lambda_2 & 0 \\ 2 & -1 - \lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \text{rref}(A - \lambda_2 I) \end{aligned}$$

Combination and multiply.

Reduced echelon form.

The partial derivative  $\partial_{t_1} \mathbf{v}$  of the general solution  $x = 0, y = t_1$  is eigenvector  $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

## 2 Example (Computing $3 \times 3$ Eigenpairs)

Find all eigenpairs of the  $3 \times 3$  matrix  $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ .

### College Algebra

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The eigenvalues are  $\lambda_1 = 1 + 2i$ ,  $\lambda_2 = 1 - 2i$ ,  $\lambda_3 = 3$ . Details:

$$0 = \det(A - \lambda I)$$

Characteristic equation.

$$= \begin{vmatrix} 1 - \lambda & 2 & 0 \\ -2 & 1 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{vmatrix}$$

Subtract  $\lambda$  from the diagonal.

$$= ((1 - \lambda)^2 + 4)(3 - \lambda)$$

Cofactor rule and Sarrus' rule.

Root  $\lambda = 3$  is found from the factored form above. The roots  $\lambda = 1 \pm 2i$  are found from the quadratic formula after expanding  $(1 - \lambda)^2 + 4 = 0$ . Alternatively, take roots across  $(\lambda - 1)^2 = -4$ .

## Linear Algebra

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The eigenpairs are

$$\left( 1 + 2i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \right), \left( 1 - 2i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \right), \left( 3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

Details appear below.



## Eigenvector $\mathbf{v}_1$ for $\lambda_1 = 1 + 2i$

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$$B = A - \lambda_1 I$$

$$= \begin{pmatrix} 1 - \lambda_1 & 2 & 0 \\ -2 & 1 - \lambda_1 & 0 \\ 0 & 0 & 3 - \lambda_1 \end{pmatrix}$$

$$= \begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \\ 0 & 0 & 2 - 2i \end{pmatrix}$$

$$\approx \begin{pmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiply rule.

$$\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Combination, factor= $-i$ .

$$\approx \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Swap rule.

$$= \text{rref}(A - \lambda_1 I)$$

Reduced echelon form.

The partial derivative  $\partial_{t_1} \mathbf{v}$  of the general solution  $x = -it_1$ ,  $y = t_1$ ,  $z = 0$  is eigenvector  $\mathbf{v}_1 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$ .

**Eigenvector  $\mathbf{v}_2$  for  $\lambda_2 = 1 - 2i$**

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The problem  $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{v}_2 = \mathbf{0}$  has solution  $\mathbf{v}_2 = \overline{\mathbf{v}_1}$ .

To see why, take conjugates across the equation to give  $(\overline{\mathbf{A}} - \overline{\lambda_2} \mathbf{I})\overline{\mathbf{v}_2} = \mathbf{0}$ . Then  $\overline{\mathbf{A}} = \mathbf{A}$  ( $\mathbf{A}$  is real) and  $\lambda_1 = \overline{\lambda_2}$  gives  $(\mathbf{A} - \lambda_1 \mathbf{I})\overline{\mathbf{v}_2} = \mathbf{0}$ . Then  $\overline{\mathbf{v}_2} = \mathbf{v}_1$ .

Finally,

$$\mathbf{v}_2 = \overline{\overline{\mathbf{v}_2}} = \overline{\mathbf{v}_1} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}.$$

## Eigenvector $v_3$ for $\lambda_3 = 3$

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$$\begin{aligned} A - \lambda_3 I &= \begin{pmatrix} 1 - \lambda_3 & 2 & 0 \\ -2 & 1 - \lambda_3 & 0 \\ 0 & 0 & 3 - \lambda_3 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \text{rref}(A - \lambda_3 I) \end{aligned}$$

Multiply rule.

Combination and multiply.

Reduced echelon form.

The partial derivative  $\partial_{t_1} \mathbf{v}$  of the general solution  $x = 0$ ,  $y = 0$ ,  $z = t_1$  is eigenvector

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$