How to Solve Linear Differential Equations

• Definition: Base Atom, Atom
• Independence of Atoms
• Construction of the General Solution from a List of Distinct Atoms
• Euler’s Theorems
  – Euler’s Basic Theorem
  – Euler’s Multiplicity Theorem
  – A Shortcut Method
• Examples
• Main Theorems on Atoms and Linear Differential Equations
Atoms

A base atom is one of $1, e^{ax}, \cos bx, \sin bx, e^{ax} \cos bx, e^{ax} \sin bx$, with $b > 0$ and $a \neq 0$.

An atom equals $x^n$ times a base atom, where $n \geq 0$ is an integer.

Details and Remarks

- Euler’s formula $e^{i\theta} = \cos \theta + i \sin \theta$ implies that an atom is constructed from the complex expression $x^n e^{ax + ibx}$ by taking real and imaginary parts.

- The powers $1, x, x^2, \ldots, x^k$ are atoms.

- The term that makes up an atom has coefficient 1, therefore $2e^x$ is not an atom, but the 2 can be stripped off to create the atom $e^x$. Zero is not an atom. Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms $x$ and $x^2$ are indeed atoms. Terms like $-e^x, e^{-x^2}, x^{5/2} \cos x, \ln |x|$ and $x/(1 + x^2)$ are not atoms.
Independence

Linear algebra defines a list of functions $f_1, \ldots, f_k$ to be linearly independent if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x)$$

for all $x$ implies $c_1 = c_2 = \cdots = c_k = 0$.

Independence and Atoms

**Theorem 1 (Atoms are Independent)**
A list of finitely many distinct atoms is linearly independent.

**Theorem 2 (Powers are Independent)**
The list of distinct atoms $1, x, x^2, \ldots, x^k$ is linearly independent. And all of its sublists are linearly independent.
Picard’s theorem says that the homogeneous constant-coefficient linear differential equation

\[ y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y = 0 \]

has solution space \( S \) of dimension \( n \). Picard’s theorem reduces the general solution problem to finding \( n \) linearly independent solutions.

Euler’s theorem infra says that the required \( n \) independent solutions can be taken to be atoms. The theorem explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation [characteristic polynomial=left side]

\[ r^n + p_{n-1}r^{n-1} + \cdots + p_1r + p_0 = 0. \]

The Fundamental Theorem of Algebra states that there are exactly \( n \) roots \( r \), real or complex, for an \( n \)th order polynomial equation. The result implies that the characteristic equation has exactly \( n \) roots, counting multiplicities.

General Solution. Because the list of atoms constructed by Euler’s theorem has \( n \) distinct elements, then this list of independent atoms forms a basis for the general solution of the differential equation, giving

\[ y = c_1(\text{atom 1}) + \cdots + c_n(\text{atom } n). \]

Symbols \( c_1, \ldots, c_n \) are arbitrary coefficients. In particular, each atom listed is itself a solution of the differential equation.
Euler’s Basic Theorem

Theorem 3 (L. Euler)
The exponential $y = e^{r_1x}$ is a solution of a constant-coefficient linear homogeneous differential of the $n$th order if and only if $r = r_1$ is a root of the characteristic equation.

- If $r_1 = a$ is a real root, then one atom $e^{ax}$ is constructed by Euler’s Theorem.
- If $r_1 = a + ib$ is a complex root ($b > 0$), then Euler’s Theorem gives a complex solution

$$e^{r_1x} = e^{ax} \cos bx + ie^{ax} \sin bx.$$  

The real and imaginary parts of this complex solutions are real solutions of the differential equation. Therefore, one complex root $r_1 = a + ib$ produces two atoms

$$e^{ax} \cos bx, \quad e^{ax} \sin bx.$$  

The conjugate root $a - ib$ produces the same two atoms, hence it is ignored.
Euler’s Multiplicity Theorem

**Definition.** A root \( r = r_1 \) of a polynomial equation \( p(r) = 0 \) has multiplicity \( k \) provided \((r - r_1)^k\) divides \( p(r) \) but \((r - r_1)^{k+1}\) does not divide \( p(r) \). The calculus equivalent is \( p^{(j)}(r_1) = 0 \) for \( j = 0, \ldots, k - 1 \) and \( p^{(k)}(r_1) \neq 0 \).

**Theorem 4 (L. Euler)**
The expression \( y = x^k e^{r_1 x} \) is a solution of a constant-coefficient linear homogeneous differential of the \( n \)th order if and only if \((r - r_1)^{k+1}\) divides the characteristic polynomial.
A Shortcut for using Euler’s Theorems

Let root $r_1$ of the characteristic equation have multiplicity $m + 1$.

- Find a base atom from Euler’s Basic Theorem.
- Multiply the base atom by $1, x, \ldots, x^m$.
  This process constructs, from the base atom, exactly $m + 1$ atoms. The atom count $m + 1$ equals the multiplicity of root $r_1$.
- A real root $r_1 = a$ will produce one base atom $e^{ax}$, to which this process is applied.
- A complex root $r_1 = a + ib$ will produce 2 base atoms $e^{ax} \cos bx, e^{ax} \sin bx$. This process is applied to both.
Atom List Examples

1. If root \( r = -3 \) has multiplicity 4, then the atom list is

\[
e^{-3x}, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}.
\]

The list is constructed by multiplying the base atom \( e^{-3x} \) by powers 1, \( x \), \( x^2 \), \( x^3 \). The multiplicity 4 of the root equals the number of constructed atoms.

2. If \( r = -3 + 2i \) is a root of the characteristic equation, then the base atoms for this root (both \( -3 + 2i \) and \( -3 - 2i \) counted) are

\[
e^{-3x} \cos 2x, \quad e^{-3x} \sin 2x.
\]

If root \( r = -3 + 2i \) has multiplicity 3, then the two real atoms are multiplied by 1, \( x \), \( x^2 \) to obtain a total of 6 atoms

\[
e^{-3x} \cos 2x, \quad xe^{-3x} \cos 2x, \quad x^2e^{-3x} \cos 2x,
\]
\[
e^{-3x} \sin 2x, \quad xe^{-3x} \sin 2x, \quad x^2e^{-3x} \sin 2x.
\]

The number of atoms generated for each base atom is 3, which equals the multiplicity of the root \( -3 + 2i \).
Theorem 5 (Homogeneous Solution $y_h$ and Atoms)
Linear homogeneous differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of atoms.

Theorem 6 (Particular Solution $y_p$ and Atoms)
A linear non-homogeneous differential equation with constant coefficients $a$ having forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

Theorem 7 (General Solution $y$ and Atoms)
A linear non-homogeneous differential equation with constant coefficients having forcing term $f(x)$ equal to a linear combination of atoms has general solution $y(x) = y_h(x) + y_p(x)$ which is a linear combination of atoms.

Proofs
The first theorem follows from Picard’s theorem, Euler’s theorem and independence of atoms. The second follows from the method of undetermined coefficients, infra. The third theorem follows from the first two.