KEY

Differential Equations 2280
Midterm Exam 2
Wednesday, 1 April 2009

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books.

Do four problems only. The problems should be those that you did not do on 25 March 2009.

No answer check is expected. Details count 75%. The answer counts 25%.

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Name. KEY S2009 Midterm 2, 2280 8:35

1. **(ch3)**
Using Euler's theorem on atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c) and (d).

(a) [25%] Find a homogeneous constant-coefficient differential equation with solutions $2e^{-x}$, $e^{-x} - e^{2x/3}$, $e^{-x} + e^{x}$.

(b) [25%] Solve $y^{(6)} + 2y^{(5)} + 5y^{(4)} = 0$.

(c) [25%] Given characteristic equation $r(r+4)(r^3+4r^2)^3(r^2+4r+5) = 0$, solve the differential equation.

(d) [25%] Given $4x''(t) + 12x'(t) + 73x(t) = 0$, which represents an unforced damped spring-mass system with $m = 4$, $c = 12$, $k = 73$. Solve the differential equation [15%]. Classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a drawing of the physical model the meaning of constants m, c, k [5%].

(a) roots $= -1, 2/3, 1$ char eq $(r+1)(r-1)(r-2/3)=0$

$(r^2-1)(3r-2)=0$

$3r^3-2r^2-3r+2=0$ $\boxed{3y'''-2y''-3y'+2y=0}$

(b) $r^4(r^2+2r+5)=0$ $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{-x}\cos 2x + c_6 e^{-x}\sin 2x$

$=$ linear combination of 6 atoms

(c) $r\, r^6 (r+4)^{3+1}((r+2)^2+1) = 0$

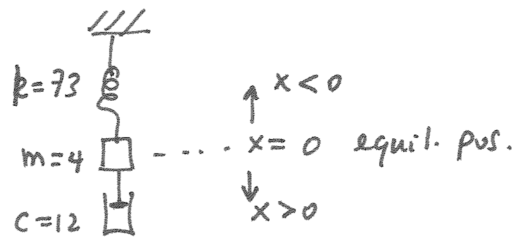
atoms $= 1, x, x^2, x^3, x^4, x^5, x^6, e^{-4x}, xe^{-4x}, x^2e^{-4x}, x^3e^{-4x}$

$e^{-2x}\cos x$, $e^{-2x}\sin x$

$y =$ linear comb. of above 13 atoms

(d) $(2r+3+8i)(2r+3-8i) = 0$ under-damped

$x = c_1 e^{-3t/2}\cos 4t + c_2 e^{-3t/2}\sin 4t$



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Name. KEY

S2009 Midterm 2, 2280 8:35

2. (ch3)
(a) [25%] The trial solution y with fewest atoms, according to the method of undetermined coefficients, contains no solution of the homogeneous equation. Explain why $y = d_1x^2 + d_2x^3$ is a trial solution with fewest atoms for $y'' = x$.

(b) [75%] Determine for $y^{(4)} + 4y^{(2)} = x^3 + 2e^{4x} + 5\sin 2x$ the corrected trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients! The trial solution should be the one with fewest atoms.

(a) $g(x) = x^2 f(x) = x^3 \Rightarrow$ atoms $1, x, x^2, x^3$ for a trial solution. Because $1, x$ are atoms for $y''=0$, then they are removed and $y = d_1x^2 + d_2x^3$ is the shortest trial solution.

(b) kill left, add right — once again

$$y = (\underbrace{\cancel{d_1} + \cancel{d_2 x}}_{\text{homog sol}} + d_3x^2 + d_4x^3 + d_5x^4 + d_6x^5)$$

kill left, add right — again

$$+ (d_7e^{4x}) + (\underbrace{d_8 \cancel{\cos} 2x + d_9 \cancel{\sin} 2x}_{\text{homog so}} + d_{10}x\cos 2x + d_{11}x\sin 2x)$$

$$y = d_3x^2 + d_4x^3 + d_5x^4 + d_6x^5$$
$$+ d_7e^{4x}$$
$$+ d_{10}x\cos 2x + d_{11}x\sin 2x$$
$$= \text{l.c. of } x^2, x^3, x^4, x^5, e^{4x}, x\cos 2x, x\sin 2x \quad (7 \text{ atoms})$$

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Name. KEY S2009 Midterm 2, 2280 8:35

3. (ch3 and ch7)
(a) [50%] Find by any applicable method the steady-state periodic solution for the current equation $I'' + 2I' + 5I = -10\sin(t)$.
(b) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' = 1 - x$. Show all steps in variation of parameters. Check the answer by quadrature.

(a) $I = d_1 \cos t + d_2 \sin t$

$-I + 2I' + 5I = -10 \sin t$

$2I' + 4I = -10 \sin t$

$-2d_1 \sin t + 2d_2 \cos t + 4d_1 \cos t + 4d_2 \sin t = -10 \sin t$

$$\begin{cases} -2d_1 + 4d_2 = -10 \\ 4d_1 + 2d_2 = 0 \end{cases} \rightarrow \begin{pmatrix} -2 & 4 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -10 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} -10 \\ 0 \end{pmatrix} \cdot \frac{1}{-20} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\boxed{I = \cos t - 2\sin t}$

(b) $y_1 = 1$, $y_2 = x$, $W = 1$, $f = 1 - x$

$$\begin{aligned} y_p &= \left(\int -\frac{y_2 f}{W}\,dx\right) y_1 + \left(\int \frac{y_1 f}{W}\,dx\right) y_2 \\ &= \left(\int \frac{-x(1-x)}{1}\,dx\right) 1 + \left(\int \frac{1(1-x)}{1}\,dx\right) x \\ &= \int (-x + x^2)\,dx + \left(\int (1-x)\,dx\right) x \\ &= -\frac{x^2}{2} + \frac{x^3}{3} + \left(x - \frac{x^2}{2}\right) x \\ &= -\frac{x^2}{2} + \frac{x^3}{3} + x^2 - \frac{x^3}{2} \\ &= \boxed{\frac{x^2}{2} + \left(-\frac{x^3}{6}\right)} \end{aligned}$$

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Name. KEY S2009 Midterm 2, 2280 8:35

4. (ch7)

(a) [50%] Solve by Laplace's method $x'' + 2x' + x = e^t$, $x(0) = x'(0) = 0$.

(b) [25%] Assume $f(t)$ is of exponential order. Find $f(t)$ in the relation

$$\frac{d}{ds}\mathcal{L}(f(t))\Big|_{s\to(s-3)} = \mathcal{L}(f(t) - t).$$

(c) [25%] Derive an integral formula for $y(t)$ by Laplace transform methods from

$$y''(t) + 4y'(t) + 4y(t) = f(t), \quad y(0) = y'(0) = 0.$$

(a) $\mathcal{L}(x) =$ Transfer × input $= \frac{1}{(s+1)^2(s-1)} = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{c}{s-1}$

$= \mathcal{L}(ae^{-t} + bte^{-t} + ce^{t})$

$x = ae^{-t} + bte^{-t} + ce^{t}$ $\quad a = -\frac{1}{4} \quad b = -\frac{1}{2} \quad c = \frac{1}{4}$

(b) $\frac{d}{ds}\mathcal{L}(f)|_{s\to s-3} = \mathcal{L}(f(t) - t)$

$\mathcal{L}((-t)f(t))|_{s\to s-3} = \mathcal{L}(f(t) - t)$ s-diff Thm

$\mathcal{L}(e^{3t}(-t)f(t)) = \mathcal{L}(f(t) - t)$ shift Thm

Lerch ⟹

$te^{3t}f(t) + f(t) = t \implies \boxed{f(t) = \frac{t}{te^{3t}+1}}$

(c) By Transfer function results

$\mathcal{L}(y) = \frac{1}{s^2+4s+4}\mathcal{L}(f) = \frac{1}{(s+2)^2}\mathcal{L}(f) = \mathcal{L}(te^{-2t})\mathcal{L}(f)$

$= \mathcal{L}\left(\int_0^t f(t-u)\,ue^{-2u}\,du\right)$ by conv. Thm.

$y = \int_0^t f(t-u)\,ue^{-2u}\,du$ by Lerch's Thm.

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Name. KEY

S2009 Midterm 2, 2280 8:35

5. (ch7)

(a) [30%] Solve $\mathcal{L}(f(t)) = \dfrac{1}{(s^2+s)(s^2-s)}$ for $f(t)$.

(b) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2+4s+5}$.

(c) [20%] Let $u(t)$ denote the unit step. Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(u(t-1)\sin 2t)$$

(d) [30%] Compute $\mathcal{L}(e^{2t}f(t))$ for

$$f(t) = \frac{e^t - e^{-t}}{t}.$$

(a) $\mathcal{L}(f) = \dfrac{1}{s^2(s+1)(s-1)} = \mathcal{L}(a + bt + ce^{-t} + de^{t})$

$f(t) = a + bt + ce^{-t} + de^{t}$, $a=0,\ b=-1,\ c=\frac{-1}{2},\ d=\frac{1}{2}$

used Heaviside on $\dfrac{1}{u(u-1)}$, $u=s^2$

Then " " $\dfrac{1}{(s-1)(s+1)}$.

(b) $\mathcal{L}(f) = \dfrac{s+1}{(s+2)^2+1} = \dfrac{s+2}{(s+2)^2+1} + \dfrac{-1}{(s+2)^2+1} = \mathcal{L}((\cos t) - (\sin t))\big|_{s\to s+2}$

$f(t) = e^{-2t}(\cos t - \sin t)$ by shift Thm.

(c) $\mathcal{L}(f) = \mathcal{L}((-t)u(t-1)\sin 2t) \implies$ (by Lerch's Thm) $f = -u(t-1)t\sin 2t$

(d) $\mathcal{L}(f(t)t) = \mathcal{L}(e^t - e^{-t}) = \dfrac{1}{s-1} - \dfrac{1}{s+1}$

$-\dfrac{d}{ds}\mathcal{L}(f) = \dfrac{1}{s-1} - \dfrac{1}{s+1} \implies \mathcal{L}(f) = \ln\left|\dfrac{s+1}{s-1}\right| + c$

by ch1 quadrature

$c = 0$ because $\lim_{s\to\infty}\mathcal{L}(f) = 0$.

Then

$$\mathcal{L}(e^{2t}f(t)) = \mathcal{L}(f)\big|_{s\to s-2} = \boxed{\ln\left|\frac{s-2+1}{s-2-1}\right|}$$

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