

Differential Equations 2280
Midterm Exam 2
Wednesday, 25 March 2008

Instructions: This in-class exam is 15 minutes. No calculators, notes, tables or books.

Do one problem only.

No answer check is expected. Details count 75%. The answer counts 25%.

Errata: 1(b) $y^{(3)}$ should be $y^{(4)}$
4(c) Label is 4(e), should be 4(e)

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1. (ch3)

Using Euler's theorem on atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c) and (d).

(a) [25%] Find a differential equation $ay'' + by' + cy = 0$ with solutions $2e^{-x}$, $e^{-x} - e^{2x/3}$.

(b) [25%] Solve $y^{(6)} + 4y^{(5)} + 4y^{(4)} = 0$.

(c) [25%] Given characteristic equation $r(r+2)(r^3 - 4r)^3(r^2 + 2r + 5) = 0$, solve the differential equation.

(d) [25%] Given $4x''(t) + 4x'(t) + 65x(t) = 0$, which represents an unforced damped spring-mass system with $m = 4$, $c = 4$, $k = 65$. Solve the differential equation [15%]. Classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a drawing of the physical model the meaning of constants m , c , k [5%].

(a) roots = $-1, 2/3 \Rightarrow (r+1)(r-2/3) = 0 \Rightarrow (r+1)(3r-2) = 0$
 $\Rightarrow 3r^2 + r - 2 = 0 \Rightarrow 3y'' + y' - 2y = 0$

(b) $r^4(r^2 + 4r + 4) = 0$, $r = 0, 0, 0, -2, -2$
 $y = c_1 + c_2x + c_3x^2 + c_4e^{-2x} + c_5xe^{-2x}$

(c) $r(r+2)(r-2)^3(r+2)^3r^3((r+1)^2+4) = 0$
 $r^4(r+2)^4(r-2)^3((r+1)^2+4) = 0$

atoms = $1, x, x^2, x^3, e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}, e^{2x}, xe^{2x}, e^{-x}\cos 2x, e^{-x}\sin 2x$

$y =$ linear combination of the atoms above (12 atoms)

(d) $4r^2 + 4r + 65 = 0 \Rightarrow 4(r + 1/2)^2 + 64 = 0 \Rightarrow (r + 1/2)^2 + 16 = 0$
 $r = -1/2 \pm 4i$
 $y = c_1 e^{-t/2} \cos 4t + c_2 e^{-t/2} \sin 4t$
 underdamped.

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2. (ch3)

(a) [25%] The trial solution y with fewest atoms, according to the method of undetermined coefficients, contains no solution of the homogeneous equation. Explain why, using the example $y'' = 1 + x$.

(b) [75%] Determine for $y^{(4)} + y^{(2)} = x + 2e^x + 3\sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients! The trial solution should be the one with fewest atoms.

(a) The homogeneous solution $y_h = c_1 + c_2 x$ is added to y_p to obtain $y = y_h + y_p$. Therefore, y_p should be truncated to only those terms not already found in y_h .

$$(b) \quad g = x^4(x + 2e^x + 3\sin x) = x^5 + 2x^4e^x + 3x^4\sin x$$

$$\begin{aligned} \text{atoms} = & 1, x, x^2, x^3, x^4, x^5 \\ & e^x, xe^x, x^2e^x, x^3e^x, x^4e^x \\ & \cos x, x\cos x, x^2\cos x, x^3\cos x, x^4\cos x \\ & \sin x, x\sin x, x^2\sin x, x^3\sin x, x^4\sin x \end{aligned}$$

Homog. sol. contains $1, x, \cos x, \sin x$

We remove 4 from each group with the same base atom

$$\begin{aligned} \text{Revised atoms} = & x^2, x^3, \\ & e^x, \\ & x\cos x, \\ & x\sin x \end{aligned}$$

$$\text{trial } y = d_1 x^2 + d_2 x^3 + d_3 e^x + d_4 x\cos x + d_5 x\sin x$$

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3. (ch3 and ch7)

(a) [50%] Find by any applicable method the steady-state periodic solution for the equation $x'' + 2x' + 5x = 10 \cos(t)$.

(b) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' + 3y' = e^x$.

(a) $x = d_1 \cos t + d_2 \sin t$ is the unique periodic solution
 $x' = -d_1 \sin t + d_2 \cos t$
 $x'' = -d_1 \cos t - d_2 \sin t$

$$(-d_1 \cos t - d_2 \sin t) + 2(-d_1 \sin t + d_2 \cos t) + 5(d_1 \cos t + d_2 \sin t) = 10 \cos t$$

$$\begin{cases} 4d_1 + 2d_2 = 10 \\ -2d_1 + 4d_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} 4 & -2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 40 \\ 20 \end{pmatrix}$$

$$x(t) = (2) \cos t + (1) \sin t$$

(b) $y_h = c_1 + c_2 e^{-3x}$

$$y_1 = 1$$

$$y_2 = e^{-3x}$$

$$W = \begin{vmatrix} 1 & e^{-3x} \\ 0 & -3e^{-3x} \end{vmatrix} = -3e^{-3x}$$

$$f = e^x$$

$$y_p = \left(\int \frac{-y_2 f}{W} dx \right) y_1 + \left(\int \frac{y_1 f}{W} dx \right) y_2$$

$$= \left(\int \frac{-e^{-3x} e^x}{-3e^{-3x}} dx \right) \cdot 1 + \left(\int \frac{1 \cdot e^x}{-3e^{-3x}} dx \right) e^{-3x}$$

$$= \frac{1}{3} e^x + \frac{-1}{3} \frac{e^{4x}}{4} e^{-3x}$$

$$= \frac{1}{3} e^x - \frac{1}{12} e^x$$

$$= \frac{1}{4} e^x$$

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4. (ch7)

(a) [60%] Solve by Laplace's method $x'' + x = \cos 2t$, $x(0) = x'(0) = 0$.

(b) [15%] Assume $f(t)$ is of exponential order such that $\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}(f(t) - 1)$.

Find $f(t)$.

(c) [25%] Derive an integral formula for $y(t)$ by Laplace transform methods from

$$y''(t) + y'(t) = f(t), \quad y(0) = y'(0) = 0.$$

$$(a) \quad \mathcal{L}(x'') + \mathcal{L}(x) = \mathcal{L}(\cos 2t)$$

$$(s^2 + 1)\mathcal{L}(x) = \frac{s}{s^2 + 4}$$

$$\mathcal{L}(x) = \frac{s}{(s^2 + 1)(s^2 + 4)} = \frac{a + bs}{s^2 + 1} + \frac{c + ds}{s^2 + 4}$$

$$= \mathcal{L}\left(a \sin t + b \cos t + c \frac{\sin 2t}{2} + d \cos 2t\right)$$

$$x(t) = a \sin t + b \cos t + \frac{c}{2} \sin 2t + d \cos 2t$$

$$a = 0, \quad b = 1, \quad c = 0, \quad d = 1$$

$$(b) \quad \mathcal{L}(-t f(t)) = \mathcal{L}(f(t) - 1) \Rightarrow -t f(t) = f(t) - 1 \Rightarrow (t+1)f(t) = 1$$

$$f(t) = \frac{1}{1+t}$$

$$(c) \quad (s^2 + s)\mathcal{L}(y) = \mathcal{L}(f)$$

$$\mathcal{L}(y) = \frac{1}{s(s+1)} \mathcal{L}(f)$$

$$\frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1} = \mathcal{L}(1 - e^{-t})$$

$$\mathcal{L}(y) = \mathcal{L}(1 - e^{-t}) \mathcal{L}(f)$$

$$y = \int_0^t (1 - e^{-t+x}) f(x) dx$$

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5. (ch7)

(a) [40%] Solve $\mathcal{L}(f(t)) = \frac{1}{(s^2 + s)(s^2 + 4s)}$ for $f(t)$.

(b) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{s-1}{s^2 + 2s + 5}$.

(c) [20%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \frac{d}{ds} \mathcal{L}(e^t \sin 2t)$$

(d) [20%] Solve for $f(t)$ in the relation

$$\frac{d}{ds} \mathcal{L}(f) = (\mathcal{L}(\cosh 4t))|_{s \rightarrow s+3}$$

(a) $\mathcal{L}(f) = \frac{1}{s^2(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+4}$

$$= \mathcal{L}(a + bt + c e^{-t} + d e^{-4t})$$

$$f(t) = a + bt + c e^{-t} + d e^{-4t}$$

$$a = -\frac{5}{16} \quad b = \frac{1}{4} \quad c = \frac{1}{2} \quad d = -\frac{1}{48}$$

use $s = \infty$ trick
 $a + c + d = 0$
 $a = -\frac{1}{3} + \frac{1}{48}$
 $= -\frac{15}{48} = -\frac{5}{16}$

(b) $\mathcal{L}(f(t)) = \frac{s-1}{(s+1)^2 + 4} = \frac{s+1}{(s+1)^2 + 4} + \frac{-2}{(s+1)^2 + 4}$

$$= \left(\frac{s}{s^2+4} + \frac{-2}{s^2+4} \right) |_{s \rightarrow s+1}$$

$$= \mathcal{L}(\cos 2t - \sin 2t) |_{s \rightarrow s+1} = \mathcal{L}(e^{-t} \cos 2t - e^{-t} \sin 2t)$$

$$f(t) = e^{-t} \cos 2t - e^{-t} \sin 2t$$

(c) $f(t) = -t e^t \sin 2t$

(d) $-t f(t) = e^{-3t} \cosh(4t)$

$$f(t) = \frac{-e^{-3t} \cosh(4t)}{t}$$

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