Differential Equations 2280
Midterm Exam 1 [8:35]
Wednesday, 25 February 2009

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

(a) [25%] Solve $y' = \frac{3 + x^2}{1 + x^2}$.

(b) [25%] Solve $y' = (2 \sin x + \cos x)(\sin x - 2 \cos x)$.

(c) [25%] Solve $y' = \frac{x \tan (\ln(1 + x^2))}{1 + x^2}$, $y(0) = 2$.

(d) [25%] Find the position $x(t)$ from the velocity model $\frac{dv}{dt}(t^2 v(t)) = 0$, $v(2) = 10$ and the position model $\frac{dx}{dt} = v(t)$, $x(2) = -20$.

(a) $y' = \frac{3 + x^2}{1 + x^2} \Rightarrow \int y' \, dx = \int \frac{3 + x^2}{1 + x^2} \, dx \Rightarrow y = \int \frac{2 \cdot 2 + 3 + x^2}{1 + x^2} \, dx \Rightarrow y = \int \frac{2 + 1 + x^2}{1 + x^2} \, dx \Rightarrow$

$y = \int 1 + \frac{2 - 1}{1 + x^2} \, dx$, $x = \tan \theta \Rightarrow y = \int 1 \, dx + \int \frac{2 \sec^2 \theta}{\sec^2 \theta} \, d\theta \Rightarrow y = x + 2 \tan^{-1} x + C$

(b) $y' = (2 \sin x + \cos x)(\sin x - 2 \cos x) \Rightarrow \int y' \, dx = \int (2 \sin x + \cos x)(\sin x - 2 \cos x) \, du = \int (\sin x - 2 \cos x \, du \Rightarrow y = \int udv \Rightarrow y = \frac{u^2}{2} + C$

$y = \frac{2 \sin x - 2 \cos x}{x} + C$

(c) $y' = \frac{x \tan (\ln(1 + x^2))}{1 + x^2}$, $du = \frac{2x}{1 + x^2} \Rightarrow \int y' \, dx = \int \frac{1}{2} \tan u \, du \Rightarrow y = \frac{1}{2} \ln \cos u + C$

$Y = \int \cos (\ln(1 + x^2)) \, dt \Rightarrow \int \cos (\ln(1 + x^2)) \, dt + C$ \Rightarrow $Y = \frac{1}{2} \ln \cos u + C$

(d) $\frac{dv}{dt}(t^2 v(t)) = 0 \Rightarrow \int \frac{dv}{dt} \, dt = \int t^2 v(t) \, dt \Rightarrow t^2 v(t) = C \Rightarrow (t^2)(t) = C \Rightarrow \frac{dx}{dt} = \frac{x(t)}{0} \Rightarrow \int \frac{dx}{dt} = \int \frac{x(t)}{0} \, dt \Rightarrow x(t) = \frac{-40}{t} + C \Rightarrow -20 = \frac{-40}{2} + C \Rightarrow C = 0$

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (Classification of Equations)

The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions $F$ and $G$.

(a) [40%] Check the problems that can be put into separable form. No details expected.

- $y' + xy = y(2x + e^x) + x^2y$
- $y' = (x - 1)(y + 1) + (1 - x)y$
- $y' = 2e^{2x} - ye^{3y} + 3e^{3x} + 2y$
- $y' + x^2e^y = xy$

(b) [10%] Is $y' + x(y + 1) = ye^x + x$ separable? No details expected.

(c) [10%] Give an example of $y' = f(x, y)$ which is separable and linear but not quadrature. No details expected.

(d) [40%] Apply tests to show that $y' = x + e^y$ is not separable and not linear. Supply all details.

\[ y' = ye^x + x - xy - x = ye^x - xy = y(e^x - x). \] Yes, separable.

(c) For example, $y' = xy$

(d) take $f(x, y) = x + e^y$, $\frac{dy}{dt} = e^y$ not independent of $y \Rightarrow$ not linear

Now $\frac{dy}{dt} = \frac{e^y}{x + e^y}$ is not independent of $x \Rightarrow$ not separable

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3. (Solve a Separable Equation)

Given \((x + 3)(y + 1)y' = ((x + 3)e^{-x^2} + 3x^2 + 2)(y - 1)(y + 2)\),

Find a non-equilibrium solution in implicit form.
To save time, do not solve for \(y\) explicitly and do not solve for equilibrium solutions.

\[
\frac{(y+1)}{(y-1)(y+2)} = \frac{(x+3)e^{-x^2} + 3x^2 + 2}{x+3} \Rightarrow \int \frac{(y+1)}{(y-1)(y+2)} \, dy = \int \frac{(x+3)e^{-x^2} + 3x^2 + 2}{x+3} \, dx
\]

\[
\int \frac{(y+1)}{(y-1)(y+2)} \, dy = \int \frac{A}{y-1} + \frac{B}{y+2} \, dy = \int \frac{\frac{2}{3}}{y-1} + \frac{\frac{1}{3}}{y+2} \, dy = \frac{2}{3} \ln |y-1| + \frac{1}{3} \ln |y+2| + C_1
\]

\[
\int \frac{(x+3)e^{-x^2} + 3x^2 + 2}{x+3} \, dx = \int \left( e^{-x^2} + \frac{3x^2}{x+3} + \frac{2}{x+3} \right) \, dx
\]

\[
\Rightarrow \int \left( e^{-x^2} + 3x - 7 + \frac{29}{x+3} \right) \, dx = -e^{-x^2} + \frac{3x^2}{2} - 9x + 29 \ln |x+3| + C_2
\]

\[
\frac{2}{3} \ln |y-1| + \frac{1}{3} \ln |y+2| = -e^{-x^2} + \frac{3x^2}{2} - 9x + 29 \ln |x+3| + C
\]

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4. (Linear Equations)

(a) [60%] Solve the linear model \( 5x'(t) = -160 + \frac{25}{2t+3} x(t) \), \( x(0) = 32 \). Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation \( \frac{dy}{dx} - (2x)y = 0 \).

(c) [20%] Solve \( 5 \frac{dy}{dx} + 10y = 7 \) using the superposition principle \( y = y_h + y_p \). Expected are answers for \( y_h \) and \( y_p \).

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(a) \( 5x'(t) = -160 + \frac{25}{2t+3} x(t) \) \( \Rightarrow \) \( x'(t) - \frac{5}{2t+3} x(t) = -32 \), \[ y = e^{-\frac{5}{2(t+3)}} x(t) \] \( \Rightarrow \) \[ y = e^{-\frac{5}{2(t+3)}} \]

\[ \left( \frac{2(t+3)}{5} \right) x(t) \]

\[ \frac{2(t+3)}{5} \]

\[ \frac{32}{3} \]

\( \Rightarrow \)

\[ x(t) = \frac{32}{3} \frac{1}{2(t+3)} \]

(b) \( \frac{dy}{dx} - 2xy = 0 \) \( \Rightarrow \) \( \frac{dy}{y} = \frac{2x}{x} dx \) \( \Rightarrow \) \( y = e^{-x^2} \)

(c) \( 5 \frac{dy}{dx} + 10y = 7 \) \( \Rightarrow \) \( y_p = \frac{7}{10} \)

\[ \frac{dy}{dx} + 2y = 0 \] \( \Rightarrow \) \( y = e^{-x} \), \( y_h = \frac{C}{e^{2x}} \)

\[ y = y_h + y_p = \frac{C}{e^{2x}} + \frac{7}{10} \]

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

\[ \frac{dx}{dt} = \left( \ln(1 + 5x^2) \right)^{1/5} (|2x - 1| - 3)^3 (2 + x)^2 (4 - x^2) (1 - x^2)^3 e^{\cos x}. \]

Expected in the phase line diagram are equilibrium points and signs of \( dx/dt \).

(b) [50%] Assume an autonomous equation \( x'(t) = f(x(t)) \). Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.