Name Simon Williams

Differential Equations 2280

Midterm Exam 1 [8:35]

Wednesday, 25 February 2009

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Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

100 **(a)** [25%] Solve $y' = \dfrac{3+x^2}{1+x^2}$.

(b) [25%] Solve $y' = (2\sin x + \cos x)(\sin x - 2\cos x)$.

(c) [25%] Solve $y' = \dfrac{x\tan\left(\ln(1+x^2)\right)}{1+x^2}$, $y(0) = 2$.

(d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(t^2 v(t)) = 0$, $v(2) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = -20$.

(a) $y' = \dfrac{3+x^2}{1+x^2} \Rightarrow \int y'\,dx = \int \dfrac{3+x^2}{1+x^2}\,dx \Rightarrow y = \int \dfrac{2-2+3+x^2}{1+x^2}\,dx \Rightarrow y = \int \dfrac{2+1+x^2}{1+x^2}\,dx \Rightarrow$

$y = \int 1 + \dfrac{2}{1+x^2}\,dx$, $\quad x = \tan\theta$, $dx = \sec^2\theta \Rightarrow y = \int 1\,dx + \int \dfrac{2\sec^2\theta}{\sec^2\theta}\,d\theta \Rightarrow \boxed{y = x + 2\tan^{-1}x + C}$

(b) $y' = (2\sin x + \cos x)(\sin x - 2\cos x) \Rightarrow$ ~~[illegible]~~

$\int y'\,dx = \int (2\sin x + \cos x)(\sin x - 2\cos x)$, $\quad u = \sin x - 2\cos x$, $du = \cos x + 2\sin x$ $\Rightarrow y = \int u\,du \Rightarrow y = \dfrac{u^2}{2} + c \Rightarrow$

$\boxed{y = \dfrac{(\sin x - 2\cos x)^2}{2} + C}$

(c) $y' = \dfrac{x\tan(\ln(1+x^2))}{1+x^2}$, $\quad u = \ln(1+x^2)$, $du = \dfrac{2x}{1+x^2}$ $\Rightarrow \int y'\,dx = \int \frac{1}{2}\tan u\,du \Rightarrow y = -\frac{1}{2}\ln|\cos u| + C$

$\cancel{y = -\frac{1}{2}\ln|\cos(\ln(1+x^2))| + C}$ $\quad y = -\frac{1}{2}\ln|\cos(\ln(1+x^2))| + c \Rightarrow 2 = -\frac{1}{2}\ln|\cos(\ln(1))| + C \Rightarrow$

$C = 2 \Rightarrow \boxed{y = -\frac{1}{2}\ln|\cos(\ln(1+x^2))| + 2}$

(d) $\dfrac{d}{dt}(t^2 v(t)) = 0 \Rightarrow \int \dfrac{d}{dt}(t^2 v(t)) = \int 0\,dt \Rightarrow t^2 v(t) = C \Rightarrow (2^2)(10) = C \Rightarrow C = 40$

$\dfrac{dx}{dt} = \dfrac{40}{t^2} \Rightarrow \int \dfrac{dx}{dt} = \int \dfrac{40}{t^2}\,dt \Rightarrow x(t) = -\dfrac{40}{t} + C \Rightarrow -20 = \dfrac{-40}{2} + C \Rightarrow C = 0$

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$\boxed{x(t) = \dfrac{-40}{t}}$

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2. **(Classification of Equations)**
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

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(a) [40%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form. No details expected.

$\boxtimes$	$y' + xy = y(2x + e^x) + x^2 y$	$\boxtimes$	$y' = (x-1)(y+1) + (1-x)y$
$\boxtimes$	$y' = 2e^{2x-y}e^{3y} + 3e^{3x+2y}$	$\square$	$y' + x^2 e^y = xy$

(b) [10%] Is $y' + x(y+1) = ye^x + x$ separable? No details expected.

(c) [10%] Give an example of $y' = f(x,y)$ which is separable and linear but not quadrature. No details expected.

(d) [40%] Apply tests to show that $y' = x + e^y$ is not separable and not linear. Supply all details.

(b) $y' = ye^x + x - xy - x = ye^x - xy = y(e^x - x)$. Yes, seperable

(c) For example, $y' = xy$

(d) take $f(x,y) = x + e^y$, $\frac{\partial y}{\partial f} = e^y$ not independent of y $\Rightarrow$ not linear

now $\dfrac{\frac{\partial y}{\partial f}}{f} = \dfrac{e^y}{x+e^y}$ is not independent of x $\Rightarrow$ not seperable

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3. (Solve a Separable Equation)
Given $(x+3)(y+1)y' = \left((x+3)e^{-x+2} + 3x^2 + 2\right)(y-1)(y+2)$.

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Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{(y+1)y'}{(y-1)(y+2)} = \frac{\left((x+3)e^{-x+2}+3x^2+2\right)}{x+3} \Rightarrow \int \frac{(y+1)\,dy}{(y-1)(y+2)} = \int \frac{\left((x+3)e^{-x+2}+3x^2+2\right)}{x+3}\,dx$$

cover up

$$\int \frac{(y+1)}{(y-1)(y+2)}dy \Rightarrow \int \frac{A}{y-1} + \frac{B}{y+2} \Rightarrow \int \frac{2/3}{y-1} + \frac{1/3}{y+2} \Rightarrow \frac{2}{3}\ln|y-1| + \frac{1}{3}\ln|y+2| + C_1$$

$$\int \frac{\left((x+3)e^{-x+2}+3x^2+2\right)}{x+3}dx \Rightarrow \int \left(e^{-x+2} + \frac{3x^2}{x+3} + \frac{2}{x+3}\right)dx,$$

$$\begin{array}{r} 3x-9 \\ x+3 \enclose{longdiv}{3x^2} \\ 3x^2+9x \\ \hline -9x \\ -9x-27 \\ \hline 27 \end{array}$$

$$\Rightarrow \int \left(e^{-x+2} + 3x - 9 + \frac{29}{x+3}\right)dx = -e^{-x+2} + \frac{3}{2}x^2 - 9x + 29\ln|x+3| + C_2$$

$$\boxed{\frac{2}{3}\ln|y-1| + \frac{1}{3}\ln|y+2| = -e^{-x+2} + \frac{3}{2}x^2 - 9x + 29\ln|x+3| + C}$$

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4. (Linear Equations)

(a) [60%] Solve the linear model $5x'(t) = -160 + \frac{25}{2t+3}x(t)$, $x(0) = 32$. Show all integrating factor steps.

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(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} - (2x)y = 0$.

(c) [20%] Solve $5\frac{dy}{dx} + 10y = 7$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $5x'(t) = -160 + \frac{25}{2t+3}x(t) \Rightarrow x'(t) - \frac{5}{2t+3}x(t) = -32$, $\rho = e^{-\int \frac{5}{2t+3}} = e^{-\frac{5}{2}\ln|2t+3|}$
$= (2t+3)^{-5/2} \Rightarrow$

$\frac{\left((2t+3)^{-5/2}x(t)\right)'}{(2t+3)^{-5/2}} = -32 \Rightarrow \int \left((2t+3)^{-5/2}x(t)\right)'dt = \int -32(2t+3)^{-5/2}dt \Rightarrow$

$(2t+3)^{-5/2}x(t) = \frac{32}{3}(2t+3)^{-3/2} + C \Rightarrow (2(0)+3)^{-5/2}(32) = \frac{32}{3}(0+3)^{-3/2} + C \Rightarrow$

$C = 3^{-5/2}\cancel{32} - \frac{3^{-3/2}}{3} = 0 \Rightarrow \boxed{x(t) = \frac{32}{3}(2t+3)}$

(b) $\frac{dy}{dx} - 2xy = 0 \Rightarrow \frac{\text{constant}}{\rho} \Rightarrow \rho = e^{\int -2x} = e^{-x^2} \Rightarrow \boxed{y = \frac{C}{e^{-x^2}}}$

(c) $5\frac{dy}{dx} + 10y = 7 \Rightarrow Y_p = \frac{7}{10}$

$5\frac{dy}{dx} + 10y = 0 \Rightarrow \frac{dy}{dx} + 2y = 0$, $\rho = e^{\int 2} = e^{2x}$, $Y_h = \frac{C}{e^{2x}}$

$$\boxed{y = Y_h + Y_p = \frac{C}{e^{2x}} + \frac{7}{10}}$$

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5. (Stability)

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(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \left(\ln(1+5x^2)\right)^{1/5} (|2x-1|-3)^3 (2+x)^2(4-x^2)(1-x^2)^3 e^{\cos x}.$$

always + always + always +

Expected in the phase line diagram are equilibrium points and signs of dx/dt.

$\ln(1+5x^2) \to 0$ @ $\ln(1)$

$x=0$

$|2x-1| = 3$, $x=2,-1$

$2+x \Rightarrow x=-2$

$4-x^2 \to x=\pm 2$

$1-x^2 \Rightarrow x=\pm 1$

$e^{\cos x} \Rightarrow$ none

$-3 \to (+)(+)(+)(-)(-)($

$-\frac{3}{2} \to (+)(+)(-)$

$-\frac{1}{2} \to (-)(+)(+)$

$\frac{1}{2} \to (-)(+)(+)$

$\frac{3}{2} \to (-)(+)(-)$

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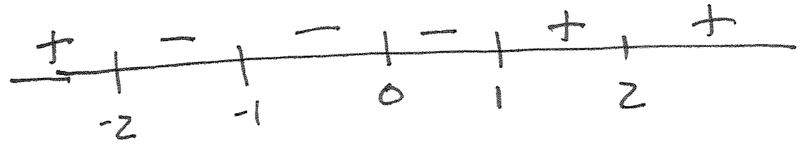
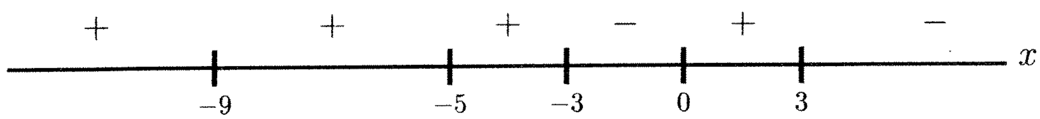
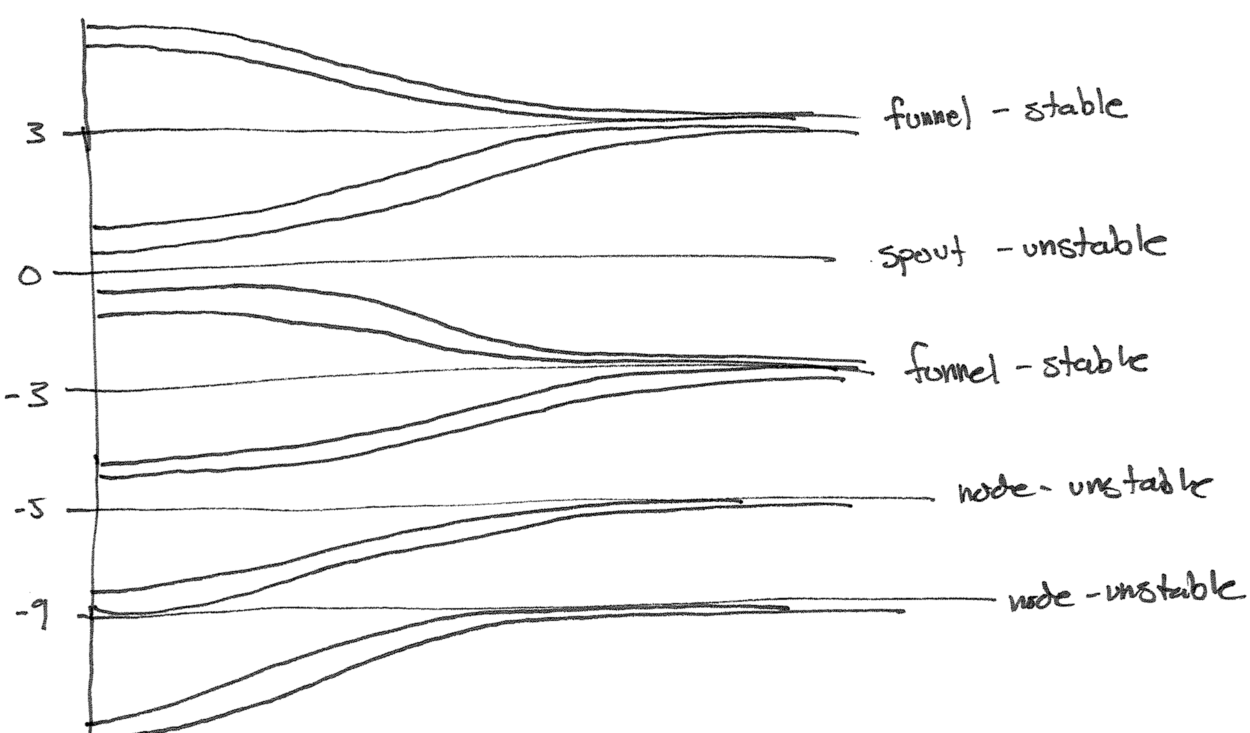


(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.





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