Systems of Differential Equations
The Eigenanalysis Method

- First Order $\mathbf{2 \times 2}$ Systems $\mathbf{x}' = \boldsymbol{A}\mathbf{x}$
- First Order $\mathbf{3 \times 3}$ Systems $\mathbf{x}' = \boldsymbol{A}\mathbf{x}$
- Second Order $\mathbf{3 \times 3}$ Systems $\mathbf{x}'' = \boldsymbol{A}\mathbf{x}$
- Vector-Matrix Form of the Solution of $\mathbf{x}' = \boldsymbol{A}\mathbf{x}$
- Four Methods for Solving a System $\mathbf{x}' = \boldsymbol{A}\mathbf{x}$

The Eigenanalysis Method for First Order 2×2 Systems

Suppose that A is 2×2 real and has eigenpairs

$$(\lambda_1, \mathbf{v}_1), \quad (\lambda_2, \mathbf{v}_2),$$

with $\mathbf{v}_1$, $\mathbf{v}_2$ independent. The eigenvalues λ_1, λ_2 can be both real. Also, they can be a complex conjugate pair $\lambda_1 = \overline{\lambda_2} = a + ib$ with $b > 0$.

Theorem 1 (Eigenanalysis Method)

The general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Solving 2×2 Systems $\mathbf{x}' = A\mathbf{x}$ with Complex Eigenvalues

If the eigenvalues are complex conjugates, then the real part $\mathbf{w}_1$ and the imaginary part $\mathbf{w}_2$ of the solution $e^{\lambda_1 t}\mathbf{v}_1$ are independent solutions of the differential equation. Then the general solution in *real form* is given by the relation

$$\mathbf{x}(t) = c_1\mathbf{w}_1(t) + c_2\mathbf{w}_2(t).$$

The Eigenanalysis Method for First Order 3×3 Systems

Suppose that A is 3×3 real and has eigenpairs

$$(\lambda_1, \mathbf{v}_1), \quad (\lambda_2, \mathbf{v}_2), \quad (\lambda_3, \mathbf{v}_3),$$

with $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ independent. The eigenvalues λ_1, λ_2, λ_3 can be all real. Also, there can be one real eigenvalue λ_3 and a complex conjugate pair of eigenvalues $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$.

Theorem 2 (Eigenanalysis Method)
The general solution of $\mathbf{x}' = A\mathbf{x}$ with 3×3 real A can be written as

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3.$$

Solving 3×3 Systems $\mathbf{x}' = A\mathbf{x}$ with Complex Eigenvalues

If there are complex eigenvalues $\lambda_1 = \overline{\lambda}_2$, then the real general solution is expressed in terms of independent solutions

$$\mathbf{w}_1 = \mathcal{R}e(e^{\lambda_1 t}\mathbf{v}_1), \ \mathbf{w}_2 = \mathcal{I}m(e^{\lambda_1 t}\mathbf{v}_1)$$

as the linear combination

$$\mathbf{x}(t) = c_1\mathbf{w}_1(t) + c_2\mathbf{w}_2(t) + c_3 e^{\lambda_3 t}\mathbf{v}_3.$$

Theorem 3 (Second Order Systems)
Let A be real and 3×3 with three negative eigenvalues $\lambda_1 = -\omega_1^2$, $\lambda_2 = -\omega_2^2$, $\lambda_3 = -\omega_3^2$. Let the eigenpairs of A be listed as

$$(\lambda_1, \mathrm{v}_1), \quad (\lambda_2, \mathrm{v}_2), \quad (\lambda_3, \mathrm{v}_3).$$

Then the general solution of the second order system $\mathrm{x}''(t) = A\mathrm{x}(t)$ is

$$\begin{aligned} \mathrm{x}(t) \; &= \; \left(a_1 \cos \omega_1 t + b_1 \frac{\sin \omega_1 t}{\omega_1}\right) \mathrm{v}_1 \\ &\quad + \left(a_2 \cos \omega_2 t + b_2 \frac{\sin \omega_2 t}{\omega_2}\right) \mathrm{v}_2 \\ &\quad + \left(a_3 \cos \omega_3 t + b_3 \frac{\sin \omega_3 t}{\omega_3}\right) \mathrm{v}_3 \end{aligned}$$

Vector-Matrix Form of the Solution of $\mathbf{x}' = A\mathbf{x}$

The solution of $\mathbf{x}' = A\mathbf{x}$ in the 3×3 case is written in vector-matrix form

$$\mathbf{x}(t) = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

This formula is normally used when the eigenpairs are real.

Complex Eigenvalues for a 2 × 2 System

When there is a complex conjugate pair of eigenvalues $\boldsymbol{\lambda_1 = \overline{\lambda}_2 = a + ib}$, $\boldsymbol{b > 0}$, then it is possible to extract a real solution $\mathbf{x}$ from the complex formula and report a real solution. The work can be organized more efficiently using the matrix product

$$\mathbf{x}(t) = e^{at}\,\mathbf{aug}(\mathcal{R}e(\mathbf{v}_1), \mathcal{I}m(\mathbf{v}_1)) \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Complex Eigenvalues for a 3×3 System

When there is a complex conjugate pair of eigenvalues $\boldsymbol{\lambda_1 = \overline{\lambda}_2 = a + ib}$, $\boldsymbol{b > 0}$, then a real solution $\mathbf{x}$ can be extracted from the complex formula to report a real solution. The work is organized using the matrix product

$$\mathbf{x}(t) \;=\; \mathbf{aug}(\boldsymbol{R}\mathrm{e}(\mathrm{v}_1), \boldsymbol{I}\mathrm{m}(\mathrm{v}_1), \mathrm{v}_3) \begin{pmatrix} e^{at}\cos bt & e^{at}\sin bt & 0 \\ -e^{at}\sin bt & e^{at}\cos bt & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Four Methods for Solving a 2×2 System $\mathbf{u}' = A\mathbf{u}$

1. **First-order method**. If A is diagonal, then use growth-decay methods. If A is triangular, then use the linear integrating factor method.
2. **Cayley-Hamilton method shortcut**. If A is not diagonal, and $a_{12} \neq 0$, then $u_1(t)$ is a linear combination of the atoms constructed from the roots r of $\det(A - rI) = 0$. Solution $u_2(t)$ is found from the system by solving for u_2 in terms of u_1 and u_1'.
3. **Eigenanalysis method**. Assume A has eigenpairs $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$ with $\mathbf{v}_1$, $\mathbf{v}_2$ independent. Then $\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$.
4. **Resolvent method**. In Laplace notation, $\mathbf{u}(t) = L^{-1}\left((sI - A)^{-1}\mathbf{u}(0)\right)$. The inverse of $C = sI - A$ is found from the formula $C^{-1} = \mathbf{adj}(C)/\det(C)$. Cramer's Rule can replace the matrix inversion method.

Four Methods for Solving an $n \times n$ System $\mathbf{u}' = A\mathbf{u}$

1. **First-order method**. If A is diagonal, then use growth-decay methods. If A is triangular, then use the linear integrating factor method.
2. **Cayley-Hamilton method**. The solution $\mathbf{u}(t)$ is a linear combination of the atoms constructed from the roots r of $\det(A - rI) = 0$,

$$\mathbf{u}(t) = (\text{atom}_1)\vec{\mathbf{d}}_1 + \cdots + (\text{atom}_n)\vec{\mathbf{d}}_n.$$

   To solve for the constant vectors $\vec{\mathbf{d}}_j$, differentiate the formula $n - 1$ times, then use $A^k\mathbf{u}(t) = \mathbf{u}^{(k+1)}(t)$ and set $t = 0$, to obtain a system for $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$.
3. **Eigenanalysis method**. Assume A has eigenpairs $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$ with $\mathbf{v}_1, \ldots, \mathbf{v}_n$ independent. Then $\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$.
4. **Resolvent method**. In Laplace notation, $\mathbf{u}(t) = L^{-1}\left((sI - A)^{-1}\mathbf{u}(0)\right)$. The inverse of $C = sI - A$ is found from the formula $C^{-1} = \mathbf{adj}(C)/\det(C)$. Cramer's Rule can replace the matrix inversion method.