Systems of Differential Equations

Matrix Methods

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Characteristic Equation

Definition 1 (Characteristic Equation)
Given a square matrix A, the **characteristic equation** of A is the polynomial equation

$$\det(A - rI) = 0.$$

The determinant $\det(A - rI)$ is formed by subtracting r from the diagonal of A. The polynomial $p(r) = \det(A - rI)$ is called the **characteristic polynomial**.

- If A is 2×2, then $p(r)$ is a quadratic.
- If A is 3×3, then $p(r)$ is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.

Characteristic Equation Examples

Create $\det(A - rI)$ by subtracting r from the diagonal of A.
Evaluate by the cofactor rule.

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2-r & 3 \\ 0 & 4-r \end{vmatrix} = (2-r)(4-r)$$

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2-r & 3 & 4 \\ 0 & 5-r & 6 \\ 0 & 0 & 7-r \end{vmatrix} = (2-r)(5-r)(7-r)$$

Cayley-Hamilton

Theorem 1 (Cayley-Hamilton)
A square matrix $\boldsymbol{A}$ satisfies its own characteristic equation.

If $\boldsymbol{p(r) = (-r)^n + a_{n-1}(-r)^{n-1} + \cdots a_0}$, then the result is the equation

$$\boldsymbol{(-A)^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0,}$$

where $\boldsymbol{I}$ is the $\boldsymbol{n \times n}$ identity matrix and $\boldsymbol{0}$ is the $\boldsymbol{n \times n}$ zero matrix.

Cayley-Hamilton Example

Assume

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{pmatrix}$$

Then

$$\boldsymbol{p(r)} = \begin{vmatrix} \mathbf{2} - \boldsymbol{r} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} - \boldsymbol{r} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} - \boldsymbol{r} \end{vmatrix} = (\mathbf{2} - \boldsymbol{r})(\mathbf{5} - \boldsymbol{r})(\mathbf{7} - \boldsymbol{r})$$

and the Cayley-Hamilton Theorem says that

$$(\mathbf{2}\boldsymbol{I} - \boldsymbol{A})(\mathbf{5}\boldsymbol{I} - \boldsymbol{A})(\mathbf{7}\boldsymbol{I} - \boldsymbol{A}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Cayley-Hamilton Method

Theorem 2 (Cayley-Hamilton Method for $\mathbf{u}' = A\mathbf{u}$)
A component function $u_k(t)$ of the vector solution $\mathbf{u}(t)$ for $\mathbf{u}'(t) = A\mathbf{u}(t)$ is a solution of the nth order linear homogeneous constant-coefficient differential equation whose characteristic equation is $\det(A - rI) = 0$.

Let $\mathbf{atom}_1, \ldots, \mathbf{atom}_n$ denote the atoms constructed from the characteristic equation $\det(A - rI) = 0$ by Euler's Theorem. Then constant vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$ exist, uniquely determined by A and $\mathbf{u}(0)$, such that

$$\mathbf{u}(t) = (\mathbf{atom}_1)\vec{\mathbf{c}}_1 + \cdots + (\mathbf{atom}_n)\vec{\mathbf{c}}_n$$

A Working Rule for Solving $\mathbf{u}' = A\mathbf{u}$

The Theorem says that $\mathbf{u}' = A\mathbf{u}$ can be solved from the formula

$$\mathbf{u}(t) = (\mathbf{atom}_1)\vec{\mathbf{c}}_1 + \cdots + (\mathbf{atom}_n)\vec{\mathbf{c}}_n$$

- The problem of solving $\mathbf{u}' = A\mathbf{u}$ is reduced to finding the vectors $\vec{\mathbf{c}}_1, \ldots, \vec{\mathbf{c}}_n$.
- The vectors $\vec{\mathbf{c}}_1, \ldots, \vec{\mathbf{c}}_n$ are **not arbitrary**, but instead **uniquely determined** by A and $\mathbf{u}(0)$!

A 2×2 Illustration

Let us solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ when A is the non-triangular matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{vmatrix} 1-r & 2 \\ 2 & 1-r \end{vmatrix} = (1-r)^2 - 4 = (r+1)(r-3).$$

Euler's theorem implies solution atoms e^{-t}, e^{3t}.

Then $\vec{\mathbf{u}}$ is a vector linear combination of the solution atoms,

$$\vec{\mathbf{u}} = e^{-t}\vec{\mathbf{c}}_1 + e^{3t}\vec{\mathbf{c}}_2.$$

Finding $\vec{\mathbf{c}}_1$ and $\vec{\mathbf{c}}_2$

To solve for $\mathbf{c}_1$ and $\mathbf{c}_2$, differentiate the above relation. Replace $\vec{\mathbf{u}}'$ by $A\vec{\mathbf{u}}$, then set $t = 0$ and $\vec{\mathbf{u}}(0) = \vec{\mathbf{u}}_0$ in the two formulas to obtain the relations

$$\begin{aligned} \vec{\mathbf{u}}_0 &= e^0\vec{\mathbf{c}}_1 + e^0\vec{\mathbf{c}}_2 \\ A\vec{\mathbf{u}}_0 &= -e^0\vec{\mathbf{c}}_1 + 3e^0\vec{\mathbf{c}}_2 \end{aligned}$$

Adding the equations gives $\vec{\mathbf{u}}_0 + A\vec{\mathbf{u}}_0 = 4\vec{\mathbf{c}}_2$ and then

$$\vec{\mathbf{c}}_1 = \frac{3}{4}\mathbf{u}_0 - \frac{1}{4}A\mathbf{u}_0, \quad \vec{\mathbf{c}}_2 = \frac{1}{4}\mathbf{u}_0 + \frac{1}{4}A\mathbf{u}_0.$$

A Matrix Method for Finding $\vec{c}_1$ and $\vec{c}_2$

The Cayley-Hamilton Method produces a unique solution for $\mathbf{c_1}$, $\mathbf{c_2}$ because the coefficient matrix

$$\begin{pmatrix} e^0 & e^0 \\ -e^0 & 3e^0 \end{pmatrix}$$

is exactly the Wronskian W of the basis of atoms evaluated at $t = 0$. This same fact applies no matter the number of coefficients $\vec{c}_1, \vec{c}_2, \ldots$ to be determined.

The answer for $\vec{c}_1$ and $\vec{c}_2$ can be written in matrix form in terms of the transpose W^T of the Wronskian matrix as

$$\text{aug}(\vec{c}_1, \vec{c}_2) = \text{aug}(\vec{u}_0, A\vec{u}_0)(W^T)^{-1}.$$

Solving a 2×2 Initial Value Problem

$$\vec{\mathbf{u}}' = A\vec{\mathbf{u}}, \quad \vec{\mathbf{u}}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $\vec{\mathbf{u}}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $A\vec{\mathbf{u}}_0 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and

$$\mathbf{aug}(\vec{\mathbf{c}}_1, \vec{\mathbf{c}}_2) = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}\left(\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}^T\right)^{-1} = \begin{pmatrix} -3/2 & 1/2 \\ 3/2 & 1/2 \end{pmatrix}.$$

The solution of the initial value problem is

$$\vec{\mathbf{u}}(t) = e^{-t}\begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix} + e^{3t}\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}e^{-t} + \frac{1}{2}e^{3t} \\ \frac{3}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix}.$$

Other Representations of the Solution $\mathbf{u}$

Let $y_1(t), \ldots, y_n(t)$ be a solution basis for the nth order linear homogeneous constant-coefficient differential equation whose characteristic equation is $\det(A - rI) = 0$.

Consider the solution basis $\mathbf{atom}_1, \mathbf{atom}_2, \ldots, \mathbf{atom}_n$. Each atom is a linear combination of $\mathbf{y}_1, \ldots, \mathbf{y}_n$. Replacing the atoms in the formula

$$\mathbf{u}(t) = (\mathbf{atom}_1)\mathbf{c}_1 + \cdots + (\mathbf{atom}_n)\mathbf{c}_n$$

by these linear combinations implies there are constant vectors $\mathbf{d}_1, \ldots, \mathbf{d}_n$ such that

$$\mathbf{u}(t) = y_1(t)\vec{\mathbf{d}}_1 + \cdots + y_n(t)\vec{\mathbf{d}}_n$$

Another General Solution of $\mathbf{u}' = A\mathbf{u}$

Theorem 3 (General Solution)
The unique solution of $\mathbf{u}' = A\mathbf{u}$, $\mathbf{u}(0) = \mathbf{u}_0$ is

$$\mathbf{u}(t) = \phi_1(t)\mathbf{u}_0 + \phi_2(t)A\mathbf{u}_0 + \cdots + \phi_n(t)A^{n-1}\mathbf{u}_0$$

where ϕ_1, ..., ϕ_n are linear combinations of atoms constructed from roots of the characteristic equation $\det(A - rI) = 0$, such that

$$\textbf{Wronskian}(\phi_1(t), \ldots, \phi_n(t))\big|_{t=0} = I.$$

Proof of the theorem

Proof: Details will be given for $n = 3$. The details for arbitrary matrix dimension n is an easy modification of this proof. The Wronskian condition implies ϕ_1, ϕ_2, ϕ_3 are independent. Then each atom constructed from the characteristic equation is a linear combination of ϕ_1, ϕ_2, ϕ_3. It follows that the unique solution $\mathbf{u}$ can be written for some vectors $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ as

$$\mathbf{u}(t) = \phi_1(t)\vec{\mathbf{d}}_1 + \phi_2(t)\vec{\mathbf{d}}_2 + \phi_3(t)\vec{\mathbf{d}}_3.$$

Differentiate this equation twice and then set $t = 0$ in all 3 equations. The relations $\mathbf{u}' = A\mathbf{u}$ and $\mathbf{u}'' = A\mathbf{u}' = AA\mathbf{u}$ imply the 3 equations

$$\begin{array}{rcccccc} \mathbf{u}_0 & = & \phi_1(0)\mathbf{d}_1 & + & \phi_2(0)\mathbf{d}_2 & + & \phi_3(0)\mathbf{d}_3 \\ A\mathbf{u}_0 & = & \phi_1'(0)\mathbf{d}_1 & + & \phi_2'(0)\mathbf{d}_2 & + & \phi_3'(0)\mathbf{d}_3 \\ A^2\mathbf{u}_0 & = & \phi_1''(0)\mathbf{d}_1 & + & \phi_2''(0)\mathbf{d}_2 & + & \phi_3''(0)\mathbf{d}_3 \end{array}$$

Because the Wronskian is the identity matrix I, then these equations reduce to

$$\begin{array}{rcccccc} \mathbf{u}_0 & = & 1\mathbf{d}_1 & + & 0\mathbf{d}_2 & + & 0\mathbf{d}_3 \\ A\mathbf{u}_0 & = & 0\mathbf{d}_1 & + & 1\mathbf{d}_2 & + & 0\mathbf{d}_3 \\ A^2\mathbf{u}_0 & = & 0\mathbf{d}_1 & + & 0\mathbf{d}_2 & + & 1\mathbf{d}_3 \end{array}$$

which implies $\mathbf{d}_1 = \mathbf{u}_0$, $\mathbf{d}_2 = A\mathbf{u}_0$, $\mathbf{d}_3 = A^2\mathbf{u}_0$.
The claimed formula for $\mathbf{u}(t)$ is established and the proof is complete.

Change of Basis Equation

Illustrated here is the change of basis formula for $n = 3$. The formula for general n is similar.

Let $\phi_1(t), \phi_2(t), \phi_3(t)$ denote the linear combinations of atoms obtained from the vector formula

$$\big(\phi_1(t), \phi_2(t), \phi_3(t)\big) = \big(\textbf{atom}_1(t), \textbf{atom}_2(t), \textbf{atom}_3(t)\big)\, C^{-1}$$

where

$$C = \textbf{Wronskian}(\textbf{atom}_1, \textbf{atom}_2, \textbf{atom}_3)(0).$$

The solutions $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$ are called the **principal solutions** of the linear homogeneous constant-coefficient differential equation constructed from the characteristic equation $\det(A - rI) = 0$. They satisfy the initial conditions

$$\textbf{Wronskian}(\phi_1, \phi_2, \phi_3)(0) = I.$$