Eigenanalysis

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What's Eigenanalysis? ──

Matrix eigenanalysis is a computational theory for the matrix equation

$$\mathbf{y} = A\mathbf{x}.$$

Fourier's Eigenanalysis Model ———————————————————————————

For exposition purposes, we assume A is a 3×3 matrix.

(1)
$$\begin{aligned}
\mathbf{x} &= c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \text{ implies} \\
\mathbf{y} &= A\mathbf{x} \\
&= c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2 + c_3 \lambda_3 \mathbf{v}_3.
\end{aligned}$$

Eigenanalysis Notation ———————————————————————————

The scale factors λ_1, λ_2, λ_3 and independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ depend only on A. Symbols c_1, c_2, c_3 stand for arbitrary numbers. This implies variable $\mathbf{x}$ exhausts all possible 3-vectors in R^3.

Fourier's Model is a Replacement Process

$$A\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3\right) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

To compute $A\mathbf{x}$ from $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, replace each vector $\mathbf{v}_i$ by its scaled version $\lambda_i\mathbf{v}_i$.

Fourier's model is said to **hold** provided there exist scale factors and independent vectors satisfying (1). Fourier's model is known to fail for certain matrices A.

Powers and Fourier's Model

Equation (1) applies to compute powers A^n of a matrix A using only the basic vector space toolkit. To illustrate, only the vector toolkit for R^3 is used in computing

$$A^5 \mathbf{x} = x_1 \lambda_1^5 \mathbf{v_1} + x_2 \lambda_2^5 \mathbf{v_2} + x_3 \lambda_3^5 \mathbf{v_3}.$$

This calculation does not depend upon finding previous powers A^2, A^3, A^4 as would be the case by using matrix multiply.

Differential Equations and Fourier's Model ————————————————————

Systems of differential equations can be solved using Fourier's model, giving a compact and elegant formula for the general solution. An example:

$$\begin{aligned} x_1' &= x_1 + 3x_2, \\ x_2' &= 2x_2 - x_3, \\ x_3' &= - 5x_3. \end{aligned}$$

The general solution is given by the formula [Fourier's theorem, proved later]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-5t} \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix},$$

which is related to Fourier's model by the symbolic formula

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + c_3 e^{\lambda_3 t} \mathbf{v}_3.$$

Fourier's model illustrated

Let

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = -5,$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.$$

Then Fourier's model holds (details later) and

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} \quad \text{implies}$$

$$A\mathbf{x} = c_1(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}$$

Eigenanalysis might be called *the method of simplifying coordinates*. The nomenclature is justified, because Fourier's model computes $\mathbf{y} = A\mathbf{x}$ by scaling independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, which is a triad or **coordinate system**.

What is Eigenanalysis?

The subject of **eigenanalysis** discovers a coordinate system $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and scale factors λ_1, λ_2, λ_3 such that Fourier's model holds. Fourier's model simplifies the matrix equation $\mathbf{y} = A\mathbf{x}$, through the formula

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

What's an Eigenvalue? ——————————————————————————————————

It is a **scale factor**. An eigenvalue is also called a *proper value* or a *hidden value* or a *characteristic value*. Symbols λ_1, λ_2, λ_3 used in Fourier's model are eigenvalues.

The **eigenvalues** of a model are scale factors. Think of them as a system of units.

What's an Eigenvector? ——————————————————————————————————

Symbols v_1, v_2, v_3 in Fourier's model are called eigenvectors, or *proper vectors* or *hidden vectors* or *characteristic vectors*. They are assumed independent.

The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues).

Data Conversion Example ───────────────────────────────────────

Let $\mathbf{x}$ in $\boldsymbol{R}^3$ be a data set variable with coordinates x_1, x_2, x_3 recorded respectively in units of meters, millimeters and centimeters. We consider the problem of conversion of the mixed-unit $\mathbf{x}$-data into proper MKS units (meters-kilogram-second) $\mathbf{y}$-data via the equations

(2)
$$\begin{aligned} y_1 &= x_1, \\ y_2 &= 0.001x_2, \\ y_3 &= 0.01x_3. \end{aligned}$$

Equations (2) are a **model** for changing units. Scaling factors $\boldsymbol{\lambda_1} = 1$, $\boldsymbol{\lambda_2} = 0.001$, $\boldsymbol{\lambda_3} = 0.01$ are the **eigenvalues** of the model.

Data Conversion Example – Continued

Problem (2) can be represented as $\mathbf{y} = A\mathbf{x}$, where the diagonal matrix A is given by

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_1 = 1, \ \lambda_2 = \frac{1}{1000}, \ \lambda_3 = \frac{1}{100}.$$

Fourier's model for this matrix A is

$$A \left(c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = c_1 \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Summary

The **eigenvalues** of a model are **scale factors**, normally represented by symbols

$$\lambda_1, \quad \lambda_2, \quad \lambda_3, \quad \ldots.$$

The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues). They are normally represented by symbols

$$v_1, \quad v_2, \quad v_3, \quad \ldots.$$

History of Fourier's Model

The subject of **eigenanalysis** was popularized by J. B. Fourier in his 1822 publication on the theory of heat, *Théorie analytique de la chaleur*. His ideas can be summarized as follows for the $n \times n$ matrix equation $\mathbf{y} = A\mathbf{x}$.

The vector $\mathbf{y} = A\mathbf{x}$ is obtained from eigenvalues λ_1, λ_2, ..., λ_n and eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ by replacing the eigenvectors by their scaled versions $\lambda_1\mathbf{v}_1$, $\lambda_2\mathbf{v}_2$, ..., $\lambda_n\mathbf{v}_n$:

$$\mathbf{x} = c_1\mathbf{v}_1 \quad + \quad c_2\mathbf{v}_2 \quad + \cdots + \quad c_n\mathbf{v}_n \quad \text{implies}$$
$$\mathbf{y} = x_1\lambda_1\mathbf{v}_1 \ + \ x_2\lambda_2\mathbf{v}_2 \ + \cdots + \ c_n\lambda_n\mathbf{v}_n.$$

Determining Equations

The eigenvalues and eigenvectors are determined by homogeneous matrix–vector equations. In Fourier's model

$$A(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3) = c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2 + c_3 \lambda_3 \mathbf{v}_3$$

choose $c_1 = 1$, $c_2 = c_3 = 0$. The equation reduces to $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$. Similarly, taking $c_1 = c_2 = 0$, $c_2 = 1$ implies $A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$. Finally, taking $c_1 = c_2 = 0$, $c_3 = 1$ implies $A\mathbf{v}_3 = \lambda_3 \mathbf{v}_3$. This proves the following fundamental result.

Theorem 1 (Determining Equations in Fourier's Model)

Assume Fourier's model holds. Then the eigenvalues and eigenvectors are determined by the three equations

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1,$$
$$A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2,$$
$$A\mathbf{v}_3 = \lambda_3 \mathbf{v}_3.$$

Determining Equations and Linear Algebra

The three relations of the theorem can be distilled into one homogeneous matrix–vector equation

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Write it as $A\mathbf{x} - \lambda\mathbf{x} = 0$, then replace $\lambda\mathbf{x}$ by $\lambda I\mathbf{x}$ to obtain the standard form[a]

$$(A - \lambda I)\mathbf{v} = 0, \quad \mathbf{v} \neq 0.$$

Let $B = A - \lambda I$. The equation $B\mathbf{v} = 0$ has a nonzero solution $\mathbf{v}$ if and only if there are infinitely many solutions. Because the matrix is square, infinitely many solutions occurs if and only if $\mathbf{rref}(B)$ has a row of zeros. Determinant theory gives a more concise statement: $\det(B) = 0$ if and only if $B\mathbf{v} = 0$ has infinitely many solutions. This proves the following result.

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[a]Identity I is required to factor out the matrix $A - \lambda I$. It is wrong to factor out $A - \lambda$, because A is 3×3 and λ is 1×1, incompatible sizes for matrix addition.

Theorem 2 (Characteristic Equation)

If Fourier's model holds, then the eigenvalues λ_1, λ_2, λ_3 are roots λ of the polynomial equation

$$\det(A - \lambda I) = 0.$$

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation**. The **characteristic polynomial** is the polynomial on the left, $\det(A - \lambda I)$, normally obtained by cofactor expansion or the triangular rule.

Eigenvectors and Frame Sequences _____

Theorem 3 (Finding Eigenvectors of A)

For each root λ of the characteristic equation, write the frame sequence for $B = A - \lambda I$ with last frame $\mathrm{rref}(B)$, followed by solving for the general solution $\mathbf{v}$ of the homogeneous equation $B\mathbf{v} = 0$. Solution $\mathbf{v}$ uses invented parameter names t_1, t_2, ... The vector basis answers $\partial_{t_1}\mathbf{v}$, $\partial_{t_2}\mathbf{v}$, ... are independent **eigenvectors** of A paired to eigenvalue λ.

Proof: The equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $B\mathbf{v} = 0$. Because $\det(B) = 0$, then this system has infinitely many solutions, which implies the frame sequence starting at B ends with $\mathrm{rref}(B)$ having at least one row of zeros. The general solution then has one or more free variables which are assigned invented symbols t_1, t_2, ..., and then the vector basis is obtained by from the corresponding list of partial derivatives. Each basis element is a nonzero solution of $A\mathbf{v} = \lambda\mathbf{v}$. By construction, the basis elements (eigenvectors for λ) are collectively independent. The proof is complete.

Eigenpairs of a Matrix ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

Definition 1 (Eigenpair)
An **eigenpair** is an eigenvalue λ together with a matching eigenvector $\mathbf{v} \neq \mathbf{0}$ satisfying the equation $A\mathbf{v} = \lambda\mathbf{v}$. The pairing implies that scale factor λ is applied to direction $\mathbf{v}$.

An applied view of an eigenpair is a coordinate axis $\mathbf{v}$ and a unit system along this axis where 1 unit is the dimensionless value λ.

A 3×3 matrix A for which Fourier's model holds has eigenvalues λ_1, λ_2, λ_3 and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. The **eigenpairs** of A are

$$(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2), (\lambda_3, \mathbf{v}_3).$$

Eigenvectors are Independent ───────────────────────────────────

Theorem 4 (Independence of Eigenvectors)

If $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ are two eigenpairs of A and $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1$, $\mathbf{v}_2$ are independent.

More generally, if $(\lambda_1, \mathbf{v}_1)$, $\ldots$, $(\lambda_k, \mathbf{v}_k)$ are eigenpairs of A corresponding to distinct eigenvalues $\lambda_!$, $\ldots$, λ_k, then $\mathbf{v}_1$, $\ldots$, $\mathbf{v}_k$ are independent.