Contents

Partial Fraction Theory

Integration theory, algebraic manipulations and Laplace theory all use partial fraction theory, which applies to polynomial fractions

$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} \tag{1}$$

where the degree of the numerator is less than the degree of the denominator.

In college algebra, it is shown that such rational functions (1) can be expressed as the sum of **partial fractions**. An example:

$$\frac{s}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{2}{s-2}.$$

Requirement: The denominators of fractions on the right must divide the denominator on the left. The numerators of fractions on the right are constants.

Definition. A rational function with constant numerator and **exactly one root** in the denominator is called a **partial fraction**.
Such terms have the form

$$\frac{A}{(s-s_0)^k}. \tag{2}$$

- The numerator in (2) is a real or complex constant A.
- The denominator in (2) has exactly one root $s = s_0$.
- The power $(s - s_0)^k$ **must divide the denominator** in the rational function (1).

Real Quadratic Partial Fractions

Assume fraction (1) has **real coefficients**. If root $s_0 = \alpha + i\beta$ in (2) is *complex*, then $(s - \overline{s_0})^k$ also divides the denominator in (1), where $\overline{s_0} = \alpha - i\beta$ is the complex conjugate of s_0. The corresponding partial fractions used in the expansion turn out to be complex conjugates of one another, which can be paired and re-written as a fraction

(3) $$\frac{A}{(s - s_0)^k} + \frac{\overline{A}}{(s - \overline{s_0})^k} = \frac{Q(s)}{((s-\alpha)^2 + \beta^2)^k},$$

where $Q(s)$ is a *real* polynomial. This justifies the replacement of all partial fractions $A/(s - s_0)^k$ with complex s_0 by

$$\frac{B + Cs}{((s - s_0)(s - \overline{s_0}))^k} = \frac{B + Cs}{((s-\alpha)^2 + \beta^2)^k},$$

in which B and C are real constants. This *real form* is preferred over the sum of complex fractions, because integral tables and Laplace tables typically contain only real formulas.

Simple Roots

Assume that (1) has *real coefficients* and the denominator of the fraction (1) has **distinct real roots** $s_1, \ldots, s_N$ and **distinct complex roots** $\alpha_1 \pm i\beta_1, \ldots, \alpha_M \pm i\beta_M$. The partial fraction expansion of (1) is a sum given in terms of *real* constants A_p, B_q, C_q by

$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} = \sum_{p=1}^{N} \frac{A_p}{s - s_p} + \sum_{q=1}^{M} \frac{B_q + C_q(s - \alpha_q)}{(s - \alpha_q)^2 + \beta_q^2}. \tag{4}$$

Multiple Roots

Assume (1) has *real coefficients* and the denominator of the fraction (1) has possibly **multiple roots**. Let N_p be the multiplicity of real root s_p and let M_q be the multiplicity of complex root $\alpha_q + i\beta_q$ ($\beta_q > 0$), $1 \le p \le N$, $1 \le q \le M$. The partial fraction expansion of (1) is given in terms of *real* constants $A_{p,k}$, $B_{q,k}$, $C_{q,k}$ by

$$\sum_{p=1}^{N} \sum_{1 \le k \le N_p} \frac{A_{p,k}}{(s - s_p)^k} + \sum_{q=1}^{M} \sum_{1 \le k \le M_q} \frac{B_{q,k} + C_{q,k}(s - \alpha_q)}{((s - \alpha_q)^2 + \beta_q^2)^k}. \tag{5}$$

Summary

The theory for simple roots and multiple roots can be distilled as follows.

> A polynomial quotient p/q with limit zero at infinity has a unique expansion into partial fractions. A partial fraction is either a constant divided by a divisor of q having exactly one real root, or else a linear function divided by a real divisor of q, having exactly one complex conjugate pair of roots.

The Sampling Method

Consider the expansion in partial fractions

$$\frac{2s-2}{s(s+1)^2(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{Ds+E}{s^2+1}. \tag{6}$$

The five undetermined real constants A through E are found by **clearing the fractions**, that is, multiply (6) by the denominator on the left to obtain the polynomial equation

$$\begin{aligned} 2s-2 \;=\; & A(s+1)^2(s^2+1) + Bs(s+1)(s^2+1) \\ & +Cs(s^2+1) + (Ds+E)s(s+1)^2. \end{aligned} \tag{7}$$

Next, five different values of s are substituted into (7) to obtain equations for the five unknowns A through E. We always use the **roots of the denominator** to start: $s = 0$, $s = -1$, $s = i$, $s = -i$ are the roots of $s(s+1)^2(s^2+1) = 0$. Each complex root results in two equations, by taking real and imaginary parts. The complex conjugate root $s = -i$ is not used, because it duplicates equations already obtained from $s = i$. The three roots $s = 0$, $s = -1$, $s = i$ give only four equations, so we invent another value $s = 1$ to get the fifth equation.

The Equations

$$\begin{array}{rcll} -2 &=& A & (s=0) \\ -4 &=& -2C & (s=-1) \\ 2i-2 &=& (Di+E)i(i+1)^2 & (s=i) \\ 0 &=& 8A+4B+2C+4(D+E) & (s=1) \end{array} \tag{8}$$

Because D and E are real, the complex equation $(s = i)$ becomes two equations, as follows.

$2i-2=(Di+E)i(i^2+2i+1)$	Expand power.
$2i-2=-2Di-2E$	Use $i^2=-1$ twice.
$2=-2D$	Equate imaginary parts.
$-2=-2E$	Equate real parts.

Solving the 5×5 system, the answers are $A = -2$, $B = 3$, $C = 2$, $D = -1$, $E = 1$.

The Method of Atoms

Consider the expansion in partial fractions

$$\frac{2s-2}{s(s+1)^2(s^2+1)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{(s+1)^2} + \frac{ds+e}{s^2+1}. \tag{9}$$

Clearing the fractions in (9)gives the polynomial equation

$$\begin{aligned} 2s-2 \;=\; & a(s+1)^2(s^2+1) + bs(s+1)(s^2+1) \\ & + cs(s^2+1) + (ds+e)s(s+1)^2. \end{aligned} \tag{10}$$

The **method of atoms** expands all polynomial products and collects on powers of s (functions 1, s, s^2, ...are called **atoms**). The coefficients of the powers are matched to give 5 equations in the five unknowns a through e. The unique solution is $a = -2$, $b = 3$, $c = 2$, $d = -1$, $e = 1$. Some details:

$$\begin{aligned} 2s-2 \;=\; & (a+b+d)\,s^4 + (2a+b+c+2d+e)\,s^3 \\ & + (2a+b+d+2e)\,s^2 + (2a+b+c+e)\,s + a \end{aligned} \tag{11}$$

Matching powers implies the equations

$$\begin{aligned} & a+b+d=0,\;\; 2a+b+c+2d+e=0,\;\; 2a+b+d+2e=0, \\ & 2a+b+c+e=2,\;\; a=-2. \end{aligned}$$

Heaviside's Coverup Method

The method applies only to the case of distinct roots of the denominator in (1). Extensions to multiple-root cases can be made. Consider the expansion

$$\frac{2s+1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}. \tag{12}$$

Mysterious Details

Oliver Heaviside proposed to find in (12) the constant $C = -\frac{1}{2}$ by a **cover–up method**:

$$\left.\frac{2s+1}{s(s-1)\boxed{}}\right|_{\boxed{s+1}=0} = \frac{C}{\boxed{}}.$$

The *instructions* are to cover–up the matching factors $(s+1)$ on the left and right with box $\boxed{}$ (Heaviside used two fingertips), then evaluate on the left at the *root* s which causes the box contents to be zero. The other terms on the right are replaced by zero.

Justifying Heaviside's Method

To begin, **clear the fraction** $C/(s+1)$, that is, multiply (12) by the denominator $\boxed{s+1}$ of the partial fraction $C/(s+1)$ to obtain the *partially-cleared fraction relation*

$$\frac{(2s+1)\boxed{(s+1)}}{s(s-1)\boxed{(s+1)}} = \frac{A\boxed{(s+1)}}{s} + \frac{B\boxed{(s+1)}}{s-1} + \frac{C\boxed{(s+1)}}{\boxed{(s+1)}}.$$

Set $\boxed{(s+1)} = 0$ in the display. Cancelations left and right plus annihilation of two terms on the right gives Heaviside's prescription

$$\left.\frac{2s+1}{s(s-1)}\right|_{s+1=0} = C.$$

The factor $(s+1)$ in (12) is by no means special: the same procedure applies to find A and B. The method works for denominators with simple roots, that is, no repeated roots are allowed.

Summary

Heaviside's method in words:

> To determine A in a given partial fraction $\frac{A}{s-s_0}$, multiply the relation by $(s - s_0)$, which partially clears the fraction. Substitute s from the equation $s - s_0 = 0$ into the partially cleared relation.

Extension to Multiple Roots

Heaviside's method can be extended to the case of repeated roots. The basic idea is to *factor–out the repeats*. To illustrate, consider the partial fraction expansion details

$$R = \frac{1}{(s+1)^2(s+2)}$$	A sample rational function having repeated roots.
$$= \frac{1}{s+1}\left(\frac{1}{(s+1)(s+2)}\right)$$	Factor–out the repeats.
$$= \frac{1}{s+1}\left(\frac{1}{s+1} + \frac{-1}{s+2}\right)$$	Apply the cover–up method to the simple root fraction.
$$= \frac{1}{(s+1)^2} + \frac{-1}{(s+1)(s+2)}$$	Multiply.
$$= \frac{1}{(s+1)^2} + \frac{-1}{s+1} + \frac{1}{s+2}$$	Apply the cover–up method to the last fraction on the right.

Terms with only one root in the denominator are already partial fractions. Thus the work centers on expansion of quotients in which the denominator has two or more roots.

Special Methods

Heaviside's method has a useful extension for the case of roots of multiplicity two. To illustrate, consider these details:

$$\begin{aligned} R &= \frac{1}{(s+1)^2(s+2)} \\ &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2} \\ &= \frac{A}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2} \\ &= \frac{-1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2} \end{aligned}$$

1. A fraction with multiple roots.
2. See equation (5), page 6.
3. Find B and C by Heaviside's cover–up method.
4. Details below.

Details

We discuss ④ details. Multiply the equation ① = ② by $s+1$ to partially clear fractions, the same step as the cover-up method:

$$\frac{1}{(s+1)(s+2)} = A + \frac{B}{s+1} + \frac{C(s+1)}{s+2}.$$

We don't substitute $s+1=0$, because it gives infinity for the second term. Instead, set $s=\infty$ to get the equation $0=A+C$. Because $C=1$ from ③, then $A=-1$.
The illustration works for one root of multiplicity two, because $s=\infty$ will resolve the coefficient not found by the cover–up method.
In general, if the denominator in (1) has a root s_0 of multiplicity k, then the partial fraction expansion contains terms

$$\frac{A_1}{s-s_0} + \frac{A_2}{(s-s_0)^2} + \cdots + \frac{A_k}{(s-s_0)^k}.$$

Heaviside's cover–up method directly finds A_k, but not A_1 to A_{k-1}.

Cover-up Method and Complex Numbers

Consider the partial fraction expansion

$$\frac{10}{(s+1)(s^2+9)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9}.$$

The symbols A, B, C are real. The value of A can be found directly by the cover-up method, giving $A = 1$. To find B and C, multiply the fraction expansion by s^2+9, in order to partially clear fractions, then formally set $s^2+9=0$ to obtain the two equations

$$\frac{10}{s+1} = Bs + C, \quad s^2+9=0.$$

Solving for B and C

The method uses the same idea used for one real root. By clearing fractions in the first, the equations become

$$10 = Bs^2 + Cs + Bs + C, \quad s^2 + 9 = 0.$$

Substitute $s^2 = -9$ into the first equation to give the linear equation

$$10 = (-9B + C) + (B + C)s.$$

Because this linear equation has two complex roots $s = \pm 3i$, then real constants B, C satisfy the 2×2 system

$$\begin{array}{rcrcr} -9B & + & C & = & 10, \\ B & + & C & = & 0. \end{array}$$

Solving gives $B = -1, C = 1$.

Extensions

The same method applies especially to fractions with 3-term denominators, like s^2+s+1. The only change made in the details is the replacement $s^2 \longrightarrow -s-1$. By repeated application of $s^2=-s-1$, the first equation can be distilled into one linear equation in s with two roots. As before, a 2×2 system results.