Examples: Solving nth Order Equations

- Atoms
- L. Euler's Theorem
- The Atom List
- First Order. Solve $2y' + 5y = 0$.
- Second Order.
  Solve $y'' + 2y' + y = 0$, $y'' + 3y' + 2y = 0$ and $y'' + 2y' + 5y = 0$.
- Third Order. Solve $y''' - y' = 0$ and $y''' - y'' = 0$.
- Fourth Order. Solve $y^{iv} - y'' = 0$.

Atoms

An **atom** is a term with coefficient 1 obtained by taking the real and imaginary parts of

$$x^j e^{ax}(\cos cx + i \sin cx), \quad j = 0, 1, 2, \ldots,$$

where a and c represent real numbers and $c \geq 0$. By definition, zero is not an atom.

Theorem 1 (L. Euler)

The function $y = x^j e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential of the nth order if and only if $(r - r_1)^{j+1}$ divides the characteristic polynomial.

Euler's theorem is used to construct solutions of the nth order differential equation. The solutions so constructed are n distinct atoms, hence independent. Picard's theorem implies the list of atoms is a basis for the solution space.

The Atom List

1. If r_1 is a real root, then the atom list for r_1 begins with $e^{r_1 x}$. The revised atom list is

$$e^{r_1 x}, xe^{r_1 x}, \ldots, x^{k-1}e^{r_1 x}$$

provided r_1 is a root of multiplicity k. This means that factor $(r - r_1)^k$ divides the characteristic polynomial, but factor $(r - r_1)^{k+1}$ does not.

2. If $r_1 = \alpha + i\beta$, with $\beta > 0$ and its conjugate $r_2 = \alpha - i\beta$ are roots of the characteristic equation, then the atom list for this pair of roots (both r_1 and r_2 counted) begins with

$$e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x.$$

For a root of multiplicity k, these real atoms are multiplied by atoms 1, ..., x^{k-1} to obtain a list of $2k$ atoms

$$\begin{array}{l} e^{\alpha x} \cos \beta x,\ \ xe^{\alpha x} \cos \beta x,\ \ \ldots,\ \ x^{k-1}e^{\alpha x} \cos \beta x, \\ e^{\alpha x} \sin \beta x,\ \ xe^{\alpha x} \sin \beta x,\ \ \ldots,\ \ x^{k-1}e^{\alpha x} \sin \beta x. \end{array}$$

1 Example (First Order) Solve $2y' + 5y = 0$ by Euler's method, verifying $y_h = c_1 e^{-5x/2}$.

Solution

$2y' + 5y = 0$	Given differential equation.
$2r + 5 = 0$	Characteristic equation. Find it by replacement $y^{(n)} \to r^n$.
$r = -5/2$	Exactly one real root.
Atom = $e^{-5x/2}$	For a real root r, the atom is e^{rx}.
$y_h = c_1 e^{-5x/2}$	The general solution y_h is written by multiplying the atom list by constants c_1, c_2, ...

2 Example (Second Order I) Solve $y'' + 2y' + y = 0$ by Euler's method, showing $y_h = c_1 e^{-x} + c_2 x e^{-x}$.

Solution

$y'' + 2y' + y = 0$	Given differential equation.
$r^2 + 2r + 1 = 0$	Characteristic equation. Use $y^{(n)} \to r^n$.
$r = -1, -1$	Exactly two real roots.
Atoms = $e^{-x},\ xe^{-x}$	For a double root r, the atom list is $e^{rx},\ xe^{rx}$.
$y_h = c_1 e^{-x} + c_2 x e^{-x}$	The general solution y_h is written by multiplying the atom list by constants $c_1, c_2, \ldots$.

3 Example (Second Order II) Solve $y'' + 3y' + 2y = 0$ by Euler's method, showing $y_h = c_1 e^{-x} + c_2 e^{-2x}$.

Solution

$y'' + 3y' + 2y = 0$	Given differential equation.
$r^2 + 3r + 2 = 0$	Characteristic equation. Use $y^{(n)} \longrightarrow r^n$.
$r = -1, -2$	Factorization $(r+2)(r+1) = 0$.
Atoms = e^{-x}, e^{-2x}	For a real root r of multiplicity one, the atom is e^{rx}.
$y_h = c_1 e^{-x} + c_2 e^{-2x}$	The general solution y_h is written by multiplying the atom list by constants $c_1, c_2, \ldots$.

4 Example (Second Order III) Solve $y''+2y'+5y = 0$ by Euler's method, showing $y_h = c_1 e^{-x}\cos 2x + c_2 x e^{-x}\sin 2x$.

Solution

$y'' + 2y' + 5y = 0$	Given differential equation.
$r^2 + 2r + 5 = 0$	Characteristic equation. Use $y^{(n)} \to r^n$.
$r = -1 + 2i, -1 - 2i$	Factorization $(r+1)^2 + 4 = 0$.
Atoms = $e^{-x}\cos 2x,\ e^{-x}\sin 2x$	For a complex root $r = \alpha + i\beta$ of multiplicity one, the atoms are $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$.
$y_h = c_1 e^{-x}\cos 2x + c_2 e^{-x}\sin 2x$	The general solution y_h is written by multiplying the atom list by constants $c_1, c_2, \ldots$.

5 Example (Third Order I) Solve $y''' - y' = 0$ by Euler's method, showing $y_h = c_1 + c_2e^x + c_3e^{-x}$.

Solution

$y''' - y' = 0$	Given differential equation.
$r^3 - r = 0$	Characteristic equation. Use $y^{(n)} \longrightarrow r^n$.
$r = 0, 1, -1$	Factorization $r(r+1)(r-1) = 0$.
Atoms = $1, e^{-x}, e^x$	For a real root r of multiplicity one, the atom is e^{rx}.
$y_h = c_1 + c_2e^{-x} + c_3e^x$	The general solution y_h is written by multiplying the atom list by constants c_1, c_2, c_3, ...

6 Example (Third Order II) Solve $y''' - y'' = 0$ by Euler's method, showing $y_h = c_1 + c_2 x + c_3 e^x$.

Solution

$y''' - y'' = 0$	Given differential equation.
$r^3 - r^2 = 0$	Characteristic equation. Use $y^{(n)} \longrightarrow r^n$.
$r = 0, 0, 1$	Factorization $r^2(r-1) = 0$.
Atoms = $1, x, e^x$	For a real root r of multiplicity one, the atom is e^{rx}.
$y_h = c_1 + c_2 x + c_3 e^x$	The general solution y_h is written by multiplying the atom list by constants c_1, c_2, c_3, ...

7 Example (Fourth Order) Solve $y^{iv} - y'' = 0$ by Euler's method, showing $y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

Solution

$y^{iv} - y'' = 0$	Given differential equation.
$r^4 - r^2 = 0$	Characteristic equation. Use $y^{(n)} \longrightarrow r^n$.
$r = 0, 0, 1, -1$	Factorization $r^2(r-1)(r+1) = 0$.
Atoms = $1, x, e^x, e^{-x}$	For a real root r of multiplicity one, the atom is e^{rx}. For a double root, the atoms are e^{rx}, xe^{rx}.
$y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$	The general solution y_h is written by multiplying the atom list by constants c_1, c_2, c_3, c_4, ...