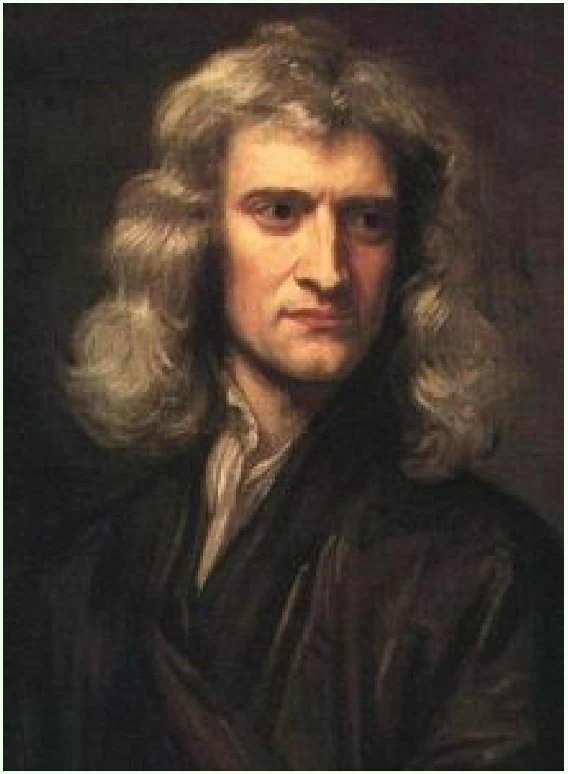
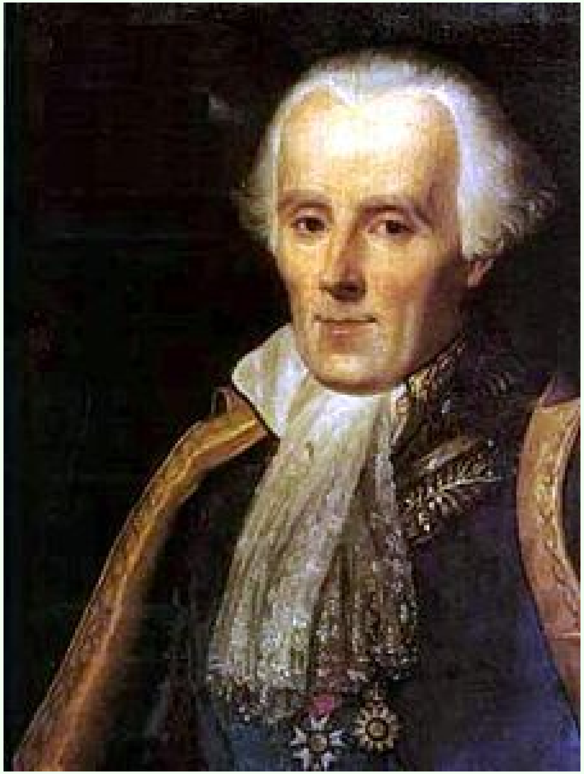
The Integral Calculus of Newton and Laplace



Isaac Newton (1643-1727) at age 46



Pierre-Simon Laplace (1749-1827)

Abstract

Presented here is a comparison of the integral calculus of Isaac Newton, normally learned in a university calculus course, against the Laplace calculus.

- The integral calculus of Isaac Newton is used extensively in science and engineering applications that assume a calculus background.
- The Laplace calculus is used in science and engineering to solve or model ordinary, integral and partial differential equations.

The reader is assumed to have some casual knowledge of both integral calculus and Laplace theory applications.

The Newton Integral Calculus

The *integral calculus* consists of rules plus an integral table.

Integral Rules	Integral Table
Fundamental Theorem	integral of a power x^n
sum rule	trigonometric integrals
constant rule	exponential integrals
parts	logarithmic integrals
u-substitution	rational functions
tabular integration	inverse trigonometric integrals
trig substitution	common radicals

- Literature expands both the set of rules and the table.
- The objective of the calculus is to compute answers to integrals and derivatives.

The Laplace Calculus

The *Laplace calculus* consists of rules plus an integral table, not of ordinary integrals, but the *Laplace integral* $\int_0^\infty f(t)e^{-st}dt$.

Laplace Rules	Laplace Integral Table
Lerch's Theorem	$\int_0^\infty (t^n)e^{-st}dt = \frac{n!}{s^{n+1}}$
sum rule	$\int_0^\infty (e^{at})e^{-st}dt = \frac{1}{s-a}$
constant rule	$\int_0^\infty (\cos bt)e^{-st}dt = \frac{s}{s+b^2}$
parts	$\int_0^\infty (\sin bt)e^{-st}dt = \frac{b}{s^2+b^2}$
shift rule	$\int_0^\infty (\mathrm{step}(t-a))e^{-st}dt = \frac{e^{-as}}{s}$
s-derivative rule	
periodic rule	
convolution rule	

- Literature expands both the set of rules and the table.
- The objective of Laplace calculus is to compute answers to Laplace integrals.

Laplace $\boldsymbol{L}$-Notation

A Laplace integral $\int_0^\infty \boldsymbol{f(t)e^{-st}dt}$ can be decomposed into

- An integral sign $\int_{\boldsymbol{E}}$ where $\boldsymbol{E}$ is the set $\boldsymbol{0 \le t < \infty}$.
- An integrand $\boldsymbol{f(t)}$.
- A Laplace integrator $\boldsymbol{dx = e^{-st}dt}$.

Symbol $\boldsymbol{L}$ replaces the integral sign $\int_{\boldsymbol{E}}$ and the Laplace integrator is omitted to obtain the definition

$$\boldsymbol{L(f(t)) \equiv \int_E f(t)dx = \int_0^\infty f(t)e^{-st}dt.}$$

A Comparison of Newton's Integral Calculus with Laplace Calculus

In the table, notation $L(f(t))$ replaces the Laplace integral $\int_0^\infty f(t)e^{-st}dt$.

Calculus Rules	Laplace Rules	Integral Table	Laplace Table
Fundamental Theorem	Lerch's Theorem	integral of a power	$L(t^n)$
sum rule	sum rule	trigonometric integrals	$L(\cos bt),\ L(\sin bt)$
constant rule	constant rule	exponential integrals	$L(e^{at})$
parts rule	parts rule	logarithmic integrals	
u-substitution	shift rules	rational functions	
tabular integration	s-derivative rule	inverse trig integrals	
trig substitution	periodic and convolution	common radicals	

- The Newton integral table is immense, while Laplace's table is extremely small.
- Laplace rules are designed to extend the table on demand.
- Both tables can be read forward and backward. This depends on the Fundamental Theorem of Calculus for Newton's calculus and on Lerch's Theorem for the Laplace calculus.

Quadrature Method

The *method of quadrature* for a quadrature differential equation

$$y' = F(x)$$

requires that you *multiply by* dx *and apply an integral sign* $\int$ *to each side*.

- The logic of the quadrature method is that equal integrands give equal integrals.
- The method works only because the fundamental theorem of calculus evaluates $\int y' dx = y + c$. This results in symbol y being isolated on the left of the equal sign.

Laplace's Method

When applied to scalar differential equations, integral equations, systems of differential equations and partial differential equations, Laplace's method is

- *Multiply the equation* by the Laplace integrator $dx = e^{-st}dt$ and apply an integral sign $\int_E$ to each side.
- Equivalently, *multiply the equation* by L, as though L was a matrix.

- Any derivatives are eliminated by the parts rule $L(y') = sL(y) - y(0)$. Then only $L(y)$ appears in the equation(s).
- Because y does not explicitly appear in the equation, but $L(y)$ does, then Lerch's theorem must be used to find y, by backwards table lookup.