Determinant Theory

- Unique Solution of $\boldsymbol{A}\mathbf{x} = \mathbf{b}$
- College Algebra Definition of Determinant
- Diagram for Sarrus' $\mathbf{3} \times \mathbf{3}$ Rule
- Transpose Rule
- How to Compute the Value of any Determinant
  - Four Rules to Compute any Determinant
  - Special Determinant Rules
  - Cofactor Expansion
- Adjugate Formula for the Inverse
- Determinant Product Theorem

Unique Solution of a 2×2 System

The 2×2 system

$$\begin{array}{rcrcl} ax & + & by & = & e, \\ cx & + & dy & = & f, \end{array} \tag{1}$$

has a unique solution provided $\Delta = ad - bc$ is nonzero, in which case the solution is given by

$$x = \frac{de - bf}{ad - bc}, \quad y = \frac{af - ce}{ad - bc}. \tag{2}$$

This result is called **Cramer's Rule** for 2×2 systems, learned in college algebra.

Determinants of Order 2

College algebra introduces matrix notation and determinant notation:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}, \quad \det(\boldsymbol{A}) = \begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{vmatrix}.$$

Evaluation of $\det(\boldsymbol{A})$ is by **Sarrus' $\mathbf{2 \times 2}$ Rule**:

$$\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{vmatrix} = \boldsymbol{ad} - \boldsymbol{bc}.$$

The first product $\boldsymbol{ad}$ is the product of the main diagonal entries and the other product $\boldsymbol{bc}$ is from the anti-diagonal.

Cramer's $\mathbf{2 \times 2}$ rule in determinant notation is

$$\boldsymbol{x} = \frac{\begin{vmatrix} \boldsymbol{e} & \boldsymbol{b} \\ \boldsymbol{f} & \boldsymbol{d} \end{vmatrix}}{\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{vmatrix}}, \quad \boldsymbol{y} = \frac{\begin{vmatrix} \boldsymbol{a} & \boldsymbol{e} \\ \boldsymbol{c} & \boldsymbol{f} \end{vmatrix}}{\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{vmatrix}}. \qquad (3)$$

Relation to Inverse Matrices

System

$$\begin{array}{rcrcl} ax & + & by & = & e, \\ cx & + & dy & = & f, \end{array} \tag{4}$$

can be expressed as the vector-matrix system $A\mathrm{u} = \mathrm{b}$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathrm{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathrm{b} = \begin{pmatrix} e \\ f \end{pmatrix}.$$

Inverse matrix theory implies

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \mathrm{u} = A^{-1}\mathrm{b} = \frac{1}{ad - bc} \begin{pmatrix} de - bf \\ af - ce \end{pmatrix}.$$

Cramer's Rule is a compact summary of the unique solution of system (4).

Unique Solution of an $n \times n$ System

System

(5)
$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \cdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

can be written as an $n \times n$ vector-matrix equation $A\vec{x} = \vec{b}$, where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{b} = (b_1, \ldots, b_n)$. The system has a unique solution provided the **determinant of coefficients** $\Delta = \det(A)$ is nonzero, and then **Cramer's Rule** for $n \times n$ systems gives

(6)
$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad \ldots, \quad x_n = \frac{\Delta_n}{\Delta}.$$

Symbol $\Delta_j = \det(B)$, where matrix B has the same columns as matrix A, except $\text{col}(B, j) = \vec{b}$.

Determinants of Order n

Determinants will be defined shortly; intuition from the $\mathbf{2} \times \mathbf{2}$ case and Sarrus' rule should suffice for the moment.

Determinant Notation for Cramer's Rule

The **determinant of coefficients** for system $\boldsymbol{A}\vec{\mathrm{x}} = \vec{\mathrm{b}}$ is denoted by

$$\boldsymbol{\Delta} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{7}$$

The other n determinants in Cramer's rule (6) are given by

$$\boldsymbol{\Delta}_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \ldots, \boldsymbol{\Delta}_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}. \tag{8}$$

College Algebra Definition of Determinant

Given an $n \times n$ matrix A, define

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\text{parity}(\sigma)} a_{1\sigma_1} \cdots a_{n\sigma_n}. \tag{9}$$

In the formula, a_{ij} denotes the element in row i and column j of the matrix A. The symbol $\sigma = (\sigma_1, \ldots, \sigma_n)$ stands for a rearrangement of the subscripts $1, 2, \ldots, n$ and S_n is the set of all possible rearrangements. The nonnegative integer parity(σ) is determined by counting the minimum number of pairwise interchanges required to assemble the list of integers $\sigma_1, \ldots, \sigma_n$ into natural order $1, \ldots, n$.

College Algebra Deinition and Sarrus' Rule

For a $\mathbf{3} \times \mathbf{3}$ matrix, the College Algebra formula reduces to **Sarrus' $\mathbf{3} \times \mathbf{3}$ Rule**

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ &\quad -a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}. \end{aligned} \tag{10}$$

Diagram for Sarrus' 3×3 Rule

The number $\det(A)$, in the 3×3 case, can be computed by the algorithm in Figure 1, which parallels the one for 2×2 matrices. The 5×3 array is made by copying the first two rows of A into rows 4 and 5.

Warning: *there is no Sarrus' rule diagram for* 4×4 *or larger matrices*!

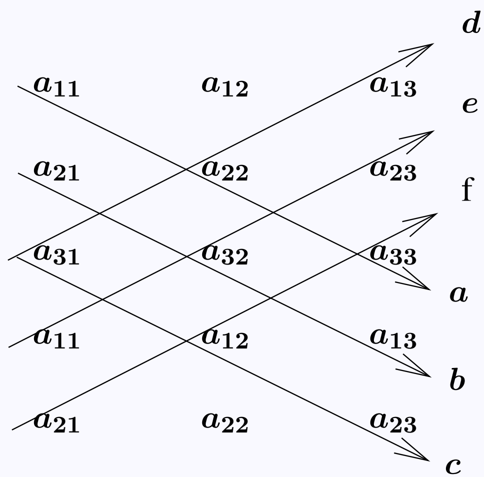

Figure 1. Sarrus' rule diagram for 3×3 matrices, which gives

$$\det(A) = (a + b + c) - (d + e + f).$$

Transpose Rule

A consequence of the college algebra definition of determinant is the relation

$$\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$$

where $\boldsymbol{A}^T$ means the transpose of $\boldsymbol{A}$, obtained by swapping rows and columns. This relation implies the following.

All determinant theory results for rows also apply to columns.

How to Compute the Value of any Determinant

- **Four Rules**. These are the *Triangular Rule*, *Combination Rule*, *Multiply Rule* and the *Swap Rule*.
- **Special Rules**. These apply to evaluate a determinant as zero.
- **Cofactor Expansion**. This is an iterative scheme which reduces computation of a determinant to a number of smaller determinants.
- **Hybrid Method**. The four rules and the cofactor expansion are combined.

Four Rules

Triangular The value of $\det(A)$ for either an upper triangular or a lower triangular matrix A is the product of the diagonal elements:

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

This is a one-arrow Sarrus' rule.

Swap If B results from A by swapping two rows, then

$$\det(A) = (-1)\det(B).$$

Combination The value of $\det(A)$ is unchanged by adding a multiple of a row to a different row.

Multiply If one row of A is multiplied by constant c to create matrix B, then

$$\det(B) = c\det(A).$$

1 Example (Four Properties) Apply the four properties of a determinant to justify the formula

$$\det\begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix} = 24.$$

Solution: Let D denote the value of the determinant. Then

$D = \det\begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix}$	Given.
$= \det\begin{pmatrix} 12 & 6 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix}$	`combo(1,2,-1)`, `combo(1,3,-1)`. Combination leaves the determinant unchanged.
$= 6\det\begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix}$	Multiply rule $m = 1/6$ on row 1 factors out a 6.
$= 6\det\begin{pmatrix} 0 & -1 & 2 \\ -1 & -1 & 1 \\ 0 & -3 & 2 \end{pmatrix}$	`combo(1,3,1)`, `combo(2,1,2)`.
$= -6\det\begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix}$	`swap(1,2)`. Swap changes the sign of the determinant.
$= 6\det\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix}$	Multiply rule $m = -1$ on row 1.
$= 6\det\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{pmatrix}$	`combo(2,3,-3)`.
$= 6(1)(-1)(-4) = 24$	Triangular rule. Formula verified.

Elementary Matrices and the Four Rules

The four rules can be stated in terms of elementary matrices as follows.

Triangular The value of $\det(A)$ for either an upper triangular or a lower triangular matrix A is the product of the diagonal elements: $\det(A) = a_{11}a_{22}\cdots a_{nn}$. This is a one-arrow Sarrus' rule valid for dimension n.

Swap If E is an elementary matrix for a swap rule, then $\det(EA) = (-1)\det(A)$.

Combination If E is an elementary matrix for a combination rule, then $\det(EA) = \det(A)$.

Multiply If E is an elementary matrix for a multiply rule with multiplier $m \neq 0$, then $\det(EA) = m\det(A)$.

Because $\det(E) = 1$ for a combination rule, $\det(E) = -1$ for a swap rule and $\det(E) = c$ for a multiply rule with multiplier $c \neq 0$, it follows that for any elementary matrix E there is the **determinant multiplication rule**

$$\det(EA) = \det(E)\det(A).$$

Special Determinant Rules

The results are stated for rows but also hold for columns, because $\det(A) = \det(A^T)$.

Zero row	If one row of A is zero, then $\det(A) = 0$.
Duplicate rows	If two rows of A are identical, then $\det(A) = 0$.
RREF $\neq I$	If $\mathrm{rref}(A) \neq I$, then $\det(A) = 0$.
Common factor	The relation $\det(A) = c\det(B)$ holds, provided A and B differ only in one row, say row j, for which $\mathrm{row}(A, j) = c\,\mathrm{row}(B, j)$.
Row linearity	The relation $\det(A) = \det(B) + \det(C)$ holds, provided A, B and C differ only in one row, say row j, for which $\mathrm{row}(A, j) = \mathrm{row}(B, j) + \mathrm{row}(C, j)$.

Cofactor Expansion for 3×3 Matrices

This is a review the college algebra topic, where the dimension of $\boldsymbol{A}$ is $\mathbf{3}$.

Cofactor row expansion means the following formulas are valid:

$$
\begin{aligned}
|A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11}(+1)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(+1)\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{21}(-1)\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}(+1)\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}(-1)\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{31}(+1)\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}(-1)\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}(+1)\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}
\end{aligned}
$$

The formulas expand a $\mathbf{3} \times \mathbf{3}$ determinant in terms of $\mathbf{2} \times \mathbf{2}$ determinants, along a row of $\boldsymbol{A}$. The attached signs ± 1 are called the **checkerboard signs**, to be defined shortly. The $\mathbf{2} \times \mathbf{2}$ determinants are called **minors** of the $\mathbf{3} \times \mathbf{3}$ determinant $|A|$. The checkerboard sign together with a minor is called a **cofactor**.

Cofactor Expansion Illustration

Cofactor expansion formulas are generally used when a row has one or two zeros, making it unnecessary to evaluate one or two of the 2×2 determinants in the expansion. To illustrate, row 1 cofactor expansion gives

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 7 \\ 5 & 4 & 8 \end{vmatrix} &= 3(+1)\begin{vmatrix} 1 & 7 \\ 4 & 8 \end{vmatrix} + 0(-1)\begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix} + 0(+1)\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 3(+1)(8-28)+0+0 \\ &= -60. \end{aligned}$$

What has been said for rows also applies to columns, due to the transpose formula

$$\det(A) = \det\left(A^T\right).$$

Minor

The $(n-1)\times(n-1)$ determinant obtained from $\det(A)$ by striking out row i and column j is called the (i,j)–minor of A and denoted $\textbf{minor}(A,i,j)$. Literature might use M_{ij} for a minor.

Cofactor

The (i,j)–cofactor of A is $\textbf{cof}(A,i,j) = (-1)^{i+j}\,\textbf{minor}(A,i,j)$.
Multiplicative factor $(-1)^{i+j}$ is called the **checkerboard sign**, because its value can be determined by counting *plus*, *minus*, *plus*, etc., from location $(1,1)$ to location (i,j) in any checkerboard fashion.

Expansion of Determinants by Cofactors

$$\det(A) = \sum_{j=1}^{n} a_{kj}\,\textbf{cof}(A,k,j), \quad \det(A) = \sum_{i=1}^{n} a_{i\ell}\,\textbf{cof}(A,i,\ell), \tag{11}$$

In (11), $1 \le k \le n$, $1 \le \ell \le n$. The first expansion is called a **cofactor row expansion** and the second is called a **cofactor column expansion**. The value $\textbf{cof}(A,i,j)$ is the cofactor of element a_{ij} in $\det(A)$, that is, the checkerboard sign times the minor of a_{ij}.

2 Example (Hybrid Method) Justify by cofactor expansion and the four properties the identity

$$\det\begin{pmatrix} 10 & 5 & 0 \\ 11 & 5 & a \\ 10 & 2 & b \end{pmatrix} = 5(6a-b).$$

Solution: Let D denote the value of the determinant. Then

$D = \det\begin{pmatrix} 10 & 5 & 0 \\ 11 & 5 & a \\ 10 & 2 & b \end{pmatrix}$	Given.
$= \det\begin{pmatrix} 10 & 5 & 0 \\ 1 & 0 & a \\ 0 & -3 & b \end{pmatrix}$	Combination leaves the determinant unchanged: `combo(1,2,-1), combo(1,3,-1).`
$= \det\begin{pmatrix} 0 & 5 & -10a \\ 1 & 0 & a \\ 0 & -3 & b \end{pmatrix}$	`combo(2,1,-10).`
$= (1)(-1)\det\begin{pmatrix} 5 & -10a \\ -3 & b \end{pmatrix}$	Cofactor expansion on column 1.
$= (1)(-1)(5b-30a)$	Sarrus' rule for $n=2$.
$= 5(6a-b).$	Formula verified.

3 Example (Cramer's Rule) Solve by Cramer's rule the system of equations

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & + & x_3 & - & x_4 & = & 1, \\ x_1 & + & x_2 & - & & & x_4 & = & -1, \\ & & 3x_2 & + & x_3 & + & x_4 & = & 3, \\ x_1 & + & & & x_3 & - & x_4 & = & 0, \end{array}$$

verifying $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$.

Solution: Form the four determinants $\mathbf{\Delta_1}, \ldots, \mathbf{\Delta_4}$ from the base determinant $\mathbf{\Delta}$ as follows:

$$\mathbf{\Delta} = \det\begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\mathbf{\Delta_1} = \det\begin{pmatrix} 1 & 3 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 3 & 3 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \mathbf{\Delta_2} = \det\begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\mathbf{\Delta_3} = \det\begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 3 & 3 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{\Delta_4} = \det\begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Five repetitions of the methods used in the previous examples give the answers $\mathbf{\Delta = -2}$, $\mathbf{\Delta_1 = -2}$, $\mathbf{\Delta_2 = 0}$, $\mathbf{\Delta_3 = -2}$, $\mathbf{\Delta_4 = -4}$, therefore Cramer's rule implies the solution

$$\boldsymbol{x_1 = \frac{\Delta_1}{\Delta}}, \quad \boldsymbol{x_2 = \frac{\Delta_2}{\Delta}}, \quad \boldsymbol{x_3 = \frac{\Delta_3}{\Delta}}, \quad \boldsymbol{x_4 = \frac{\Delta_4}{\Delta}}.$$

Then $\boldsymbol{x_1 = 1}$, $\boldsymbol{x_2 = 0}$, $\boldsymbol{x_3 = 1}$, $\boldsymbol{x_4 = 2}$.

Maple Code for Cramer’s Rule

The details of the computation above can be checked in computer algebra system `maple` as follows.

```
with(linalg):
A:=matrix([
[2, 3, 1, -1], [1, 1, 0, -1],
[0, 3, 1, 1], [1, 0, 1, -1]]);
Delta:= det(A);
b:=vector([1,-1,3,0]):
B1:=A: col(B1,1):=b:
Delta1:=det(B1);
x[1]:=Delta1/Delta;
```

The Adjugate Matrix

The **adjugate** $\textbf{adj}(A)$ of an $n \times n$ matrix A is the transpose of the matrix of cofactors,

$$\textbf{adj}(A) = \begin{pmatrix} \textbf{cof}(A,1,1) & \textbf{cof}(A,1,2) & \cdots & \textbf{cof}(A,1,n) \\ \textbf{cof}(A,2,1) & \textbf{cof}(A,2,2) & \cdots & \textbf{cof}(A,2,n) \\ \vdots & \vdots & \cdots & \vdots \\ \textbf{cof}(A,n,1) & \textbf{cof}(A,n,2) & \cdots & \textbf{cof}(A,n,n) \end{pmatrix}^T .$$

A cofactor $\textbf{cof}(A,i,j)$ is the checkerboard sign $(-1)^{i+j}$ times the corresponding minor determinant $\textbf{minor}(A,i,j)$.

Adjugate of a 2 × 2

$$\textbf{adj} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

In words: *swap the diagonal elements and change the sign of the off–diagonal elements.*

Adjugate Formula for the Inverse

For any $n \times n$ matrix

$$A \cdot \mathbf{adj}(A) = \mathbf{adj}(A) \cdot A = \det(A)\, I.$$

The equation is valid even if A is not invertible. The relation suggests several ways to find $\det(A)$ from A and $\mathbf{adj}(A)$ with one dot product.

For an invertible matrix A, the relation implies $A^{-1} = \mathbf{adj}(A)/\det(A)$:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \mathbf{cof}(A,1,1) & \mathbf{cof}(A,1,2) & \cdots & \mathbf{cof}(A,1,n) \\ \mathbf{cof}(A,2,1) & \mathbf{cof}(A,2,2) & \cdots & \mathbf{cof}(A,2,n) \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{cof}(A,n,1) & \mathbf{cof}(A,n,2) & \cdots & \mathbf{cof}(A,n,n) \end{pmatrix}^T$$

Application: Adjugate Shortcut

Given $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, then we can compute $\textbf{adj}(\boldsymbol{A}) = \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix}$.

Suppose that we mark some unknown entries in $\textbf{adj}(\boldsymbol{A})$ by ? and write $|\boldsymbol{A}|$ for $\det(\boldsymbol{A})$. Then the formula $\boldsymbol{A}\,\textbf{adj}(\boldsymbol{A}) = \textbf{adj}(\boldsymbol{A})\boldsymbol{A} = \det(\boldsymbol{A})\boldsymbol{I}$ becomes

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} ? & 3 & ? \\ ? & 1 & ? \\ ? & -1 & ? \end{pmatrix} = \begin{pmatrix} ? & 3 & ? \\ ? & 1 & ? \\ ? & -1 & ? \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} |\boldsymbol{A}| & 0 & 0 \\ 0 & |\boldsymbol{A}| & 0 \\ 0 & 0 & |\boldsymbol{A}| \end{pmatrix}.$$

While the second product $\textbf{adj}(\boldsymbol{A})\boldsymbol{A}$ contains useless information, the first product gives $\mathbf{row}(\boldsymbol{A}, 2)\,\mathbf{col}(\textbf{adj}(\boldsymbol{A}), 2) = \det(\boldsymbol{A})$. Because the values are known, then $\det(\boldsymbol{A}) = 6 + 1 + 0 = 7$.

Knowing $\boldsymbol{A}$ and $\textbf{adj}(\boldsymbol{A})$ gives the value of $\det(\boldsymbol{A})$ in one dot product.

Elementary Matrices

Theorem 1 (Determinants and Elementary Matrices)
Let E be an $n \times n$ elementary matrix. Then

Combination	$\det(E) = 1$
Multiply	$\det(E) = m$ for multiplier m.
Swap	$\det(E) = -1$
Product	$\det(EX) = \det(E)\det(X)$ for all $n \times n$ matrices X.

Theorem 2 (Determinants and Invertible Matrices)
Let A be a given invertible matrix. Then

$$\det(A) = \frac{(-1)^s}{m_1 m_2 \cdots m_r}$$

where s is the number of swap rules applied and $m_1, m_2, \ldots, m_r$ are the nonzero multipliers used in multiply rules when A is reduced to $\mathrm{rref}(A)$.

Determinant Products

Theorem 3 (Determinant Product Rule)
Let A and B be given $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B).$$

Proof

Assume one of A or B has zero determinant. Then $\det(A)\det(B) = 0$. If $\det(B) = 0$, then $B\mathrm{x} = 0$ has infinitely many solutions, in particular a nonzero solution x. Multiply $B\mathrm{x} = 0$ by A, then $AB\mathrm{x} = 0$ which implies AB is not invertible. Then the identity $\det(AB) = \det(A)\det(B)$ holds, because both sides are zero. If $\det(B) \neq 0$ but $\det(A) = 0$, then there is a nonzero y with $A\mathrm{y} = 0$. Define $\mathrm{x} = AB^{-1}\mathrm{y}$. Then $AB\mathrm{x} = A\mathrm{y} = 0$, with $\mathrm{x} \neq 0$, which implies the identity holds.. This completes the proof when one of A or B is not invertible.

Assume A, B are invertible and then $C = AB$ is invertible. In particular, $\mathrm{rref}(A^{-1}) = \mathrm{rref}(B^{-1}) = I$. Write $I = \mathrm{rref}(A^{-1}) = E_1E_2\cdots E_kA^{-1}$ and $I = \mathrm{rref}(B^{-1}) = F_1F_2\cdots F_mB^{-1}$ for elementary matrices E_i, F_j. Then

(12) $$AB = E_1E_2\cdots E_kF_1F_2\cdots F_m.$$

The theorem follows from repeated application of the basic identity $\det(EX) = \det(E)\det(X)$ to relation (12), because

$$\det(A) = \det(E_1)\cdots\det(E_k), \quad \det(B) = \det(F_1)\cdots\det(F_m).$$