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Three Examples _____

Growth-Decay
$$\frac{dA}{dt} = kA(t),$$
$$A(0) = A_0$$

$$\boxed{A(t) = A_0 e^{kt}}$$

Newton Cooling
$$\frac{du}{dt} = -h(u(t) - u_1),$$
$$u(0) = u_0$$

$$\boxed{u(t) = u_1 + (u_0 - u_1)e^{-ht}}$$

Verhulst Logistic
$$\frac{dP}{dt} = (a - bP(t))P(t),$$
$$P(0) = P_0$$

$$\boxed{P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}}$$

Theorem 1 (Peano)

Let $f(x, y)$ be continuous in a domain D of the xy-plane and let (x_0, y_0) belong to the interior of D. Then there is a small $h > 0$ and a function $y(x)$ continuously differentiable on $|x - x_0| < h$ such that $(x, y(x))$ remains in D for $|x - x_0| < h$ and $y(x)$ is one solution (many more might exist) of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Meaning _____

Given a box

$$B = \{(x, y) : |x - x_0| \leq H, |y - y_0| \leq b\}$$

with center (x_0, y_0) and f continuous on B, then there is smaller box

$$B_1 = \{(x, y) : |x - x_0| \leq h, |y - y_0| \leq b\}$$

which contains an edge-to-edge solution curve $y(x)$ of $y' = f(x, y)$ passing through the center of the box.

Definition 1 (Picard Iteration)

Define the constant function $y_0(x) = y_0$ and then define by iteration

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt.$$

The sequence $y_0(x)$, $y_1(x)$, ... is called the **sequence of Picard iterates** for $y' = f(x, y)$, $y(x_0) = y_0$.

Growth-Decay Example $y' = ky$, $y(0) = y_0$ ————————————————

The first few iterates are

$$
\begin{aligned}
y_0(x) &= y_0 \quad \text{(a constant function)} \\
y_1(x) &= y_0 + kxy_0 \\
y_2(x) &= y_0 + kxy_0 + (kx)^2 y_0/2 \\
y_3(x) &= y_0 + kxy_0 + (kx)^2 y_0/2 + (kx)^3 y_0/6.
\end{aligned}
$$

Theorem 2 (Picard-Lindelöf)

Let $f(x, y)$ and $f_y(x, y)$ be continuous in a domain D of the xy-plane. Let (x_0, y_0) belong to the interior of D. Then there is a small $h > 0$ and a *unique* function $y(x)$ continuously differentiable on $|x - x_0| < h$ such that $(x, y(x))$ remains in D for $|x - x_0| < h$ and $y(x)$ solves

$$y' = f(x, y), \quad y(x_0) = y_0.$$

The equation

$$\lim_{n \to \infty} y_n(x) = y(x)$$

is satisfied for $|x - x_0| < h$ by the Picard iterates y_n.

Explanation

Similar to the Peano Theorem, given a box B on which f and f_y are continuous, there is a smaller box B_1 and *exactly one* edge-to-edge solution curve $y(x)$ of $y' = f(x, y)$ passing through the center of the box.

Engineering translation

If f and f_y are continuous, then standard numerical methods apply to approximate the unique solution $y(x)$.

1 Example (Applying Picard's Theorem) Consider the problem $y' = 3(y-1)^{2/3}$, $y(0) = 1$. Explain why Peano's theorem applies but Picard's theorem does not.

Solution

The initial value problem is written as $y' = f(x, y)$, $y(x_0) = y_0$. Then $x_0 = 0$, $y_0 = 1$ and

$$f(x, y) = 3(y-1)^{2/3}, \quad f_y(x, y) = 2(y-1)^{-1/3}.$$

Any selected box B with center $x = 0$, $y = 1$ satisfies f continuous on B. Peano's theorem applies to conclude the initial value problem has an edge-to-edge solution in a smaller box B_1.

There is no such box B on which f_y is continuous, because f_y is undefined at $y = 1$. Picard's theorem does not apply.

Conclusion

We cannot conclude there are multiple solutions, based only on Picard's theorem. In order to conclude there are two or more solutions, we have to exhibit them. It turns out that $y = 1$ and $y = 1 + x^3$ are two solutions. Sometimes, the only signal for multiple solutions is failure of the Picard theorem hypothesis.

Direction Fields

The **method of direction fields** is a graphical method discovered by L. Euler (1707–1783) for displaying the general shape and behavior of solutions the to equation $y' = f(x, y)$ *without solving the differential equation*.

What's a Direction Field?

It is a graph window with a grid of points $P = (x_0, y_0)$ and attached vector proportional to $\vec{v}(P) = \vec{\imath} + f(x_0, y_0)\vec{\jmath}$, that fills the white space in the window.
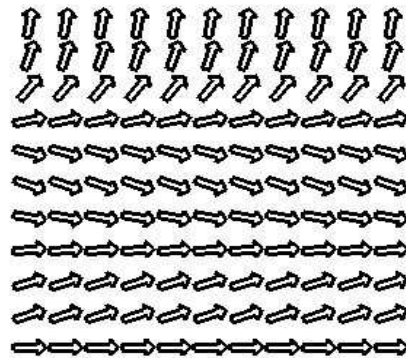

Figure 1.   A direction field for $y' = y(2 - y)(1 - y)$ on $0 \le x \le 3, 0 \le y \le 3$.

What do the arrows represent? _____

The re-scaled vector is the tangent line direction for the solution $y(x)$ of the initial value problem

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

The curve $y(x)$ has position vector $\vec{r}(x) = x\vec{i} + y(x)\vec{j}$. Its velocity vector at (x_0, y_0) is

$$\frac{d\vec{r}}{dx} = \vec{i} + y'(x_0)\vec{j} = \vec{i} + f(x_0, y_0)\vec{j}.$$

Short solution curves _____

To pixel resolution, an arrow shaft is a *short solution curve* to $y' = f(x, y)$, because the tangent line approximates the curve at the contact point.

Model Replacement

Euler's idea replaces the differential equation model $y' = f(x, y)$ by a graphical model, a direction field.

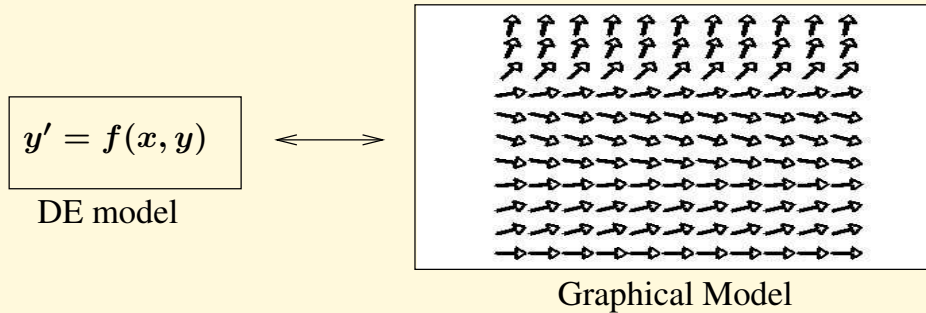$$\boxed{y' = f(x, y)}$$

DE model ⟷ Graphical Model

Figure 2. Model Replacement.

A differential equation model $y' = f(x, y)$ is replaced by a direction field model, a graphic consisting of grid points and line segments. A line segment represents a solution curve through the grid point, to pixel resolution. Model replacement means the differential equation model is tossed aside and we deal only with the graphical model.

Rules for Drawing Threaded Solutions

A direction field graphic replaces all the information supplied by the equation $y' = f(x, y)$. The equation is tossed aside and not used. *Visualization* of all solutions involves drawing a small number of edge-to-edge solutions $y(x)$ onto the direction field graph window. Used are just these rules:

1. Solutions don't cross.

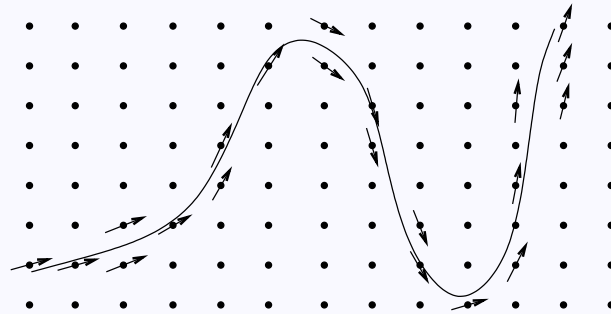**2**. Nearby tangents nearly match.

Figure 3. Threading a Solution Edge-to-Edge.

Shown is a threaded solution curve for $y' = f(x, y)$ plus nearby grid points and relevant line segments (arrows). The solution threads its way through the direction field, matching tangents at nearby grid points. Technically, the arrows cannot touch the threaded curve, unless the arrow lies atop the curve.
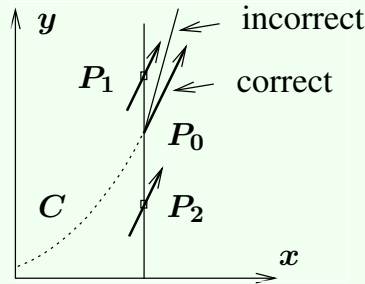
Figure 4. Threading Rules.

Solution curve C threads from the left and meets a line at point P_0. The line contains two nearby grid points P_1, P_2. The tangent at P_0 must nearly match direction field arrows at P_1, P_2.

Explanation

The slopes of the tangents in Figure 4 are given by $y' = f(x, y)$. For points $P_1 = (x_1, y_1)$, $P_0 = (x_0, y_0)$ and $P_2 = (x_2, y_2)$, the slopes are $f(x_1, y_1)$, $f(x_0, y_0)$, $f(x_2, y_2)$. If P_0, P_1, P_2 are close, then continuity of f implies all three slopes are close.

2 Example (Threading a Solution) Starting at the black dots in the direction field graphic of Figure 5, thread three solution curves.
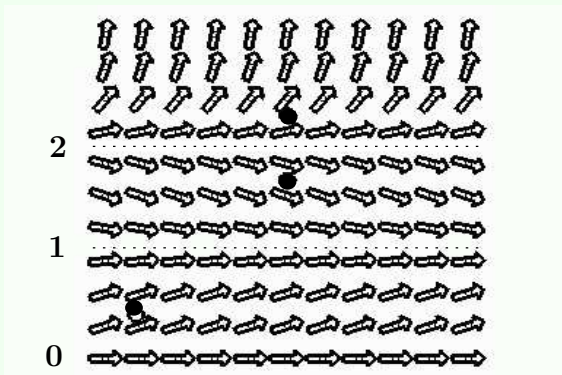

Figure 5. A direction field.

A field for the differential equation $y' = y(2 - y)(1 - y)$ is plotted on graph window $0 \leq x \leq 3$, $0 \leq y \leq 3$. The black dots are at $(0.25, 0.4)$, $(1.5, 2.25)$ and $(1.5, 1.65)$.

Solution _____

A plot appears in Figure 6. A threaded solution matches its tangents with nearby lineal elements of the direction field in Figure 5. Each threaded curve represents a solution of the differential equation through the given dot on the entire interval $0 \leq x \leq 3$, whereas the lineal elements represent solutions through the grid point on a very short x-interval.
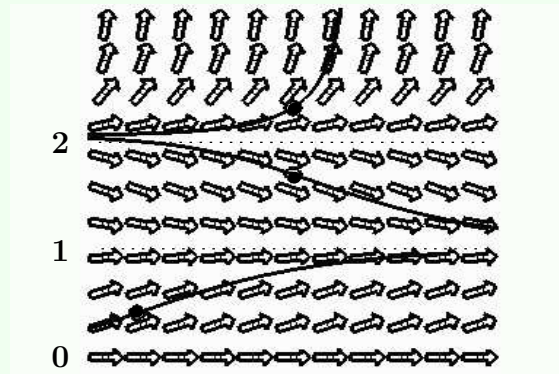

Figure 6. Threaded solutions.

The graph window is $0 \leq x \leq 3, 0 \leq y \leq 3$. Threaded curves cannot cross direction field segments nor other threaded solutions, including the constant threaded solutions $y = 0, y = 1$ and $y = 2$.