Definition

Atoms A and B are *related* if and only if their successive derivatives share a common atom. Then x^3 is related to x and x^{101}, while x is unrelated to e^x, xe^x and $x\sin x$. Atoms $x\sin x$ and $x^3\cos x$ are related, but the atoms $\cos 2x$ and $\sin x$ are unrelated.

The Basic Trial Solution Method

The method is outlined here for a second order differential equation $ay'' + by' + cy = f(x)$. The method applies unchanged for nth order equations.

Step **1**. Extract all distinct atoms from $f(x)$, $f'(x)$, $f''(x)$, ... to construct a maximal list of k atoms. Multiply these atoms by **undetermined coefficients** $d_1, d_2, \ldots, d_k$, then add, defining **trial solution** y.

Step **2**. Substitute y into the differential equation.

Fixup Rule I. If some variable d_p is missing in the substituted equation, then step 2 fails. Correct the trial solution as follows. Variable d_p appears in trial solution y as term d_pA, where A is an atom. Multiply A and all its related atoms B by x. The modified expression y is called a **corrected trial solution**. Repeat step 2 until the substituted equation contains all of the variables $d_1, \ldots, d_k$.

Step **3**. Match coefficients of atoms left and right to write out linear algebraic equations for d_1, $d_2, \ldots, d_k$. Solve the equations for the unique solution.

Step **4**. The corrected trial solution y with evaluated coefficients $d_1, d_2, \ldots, d_k$ becomes the particular solution y_p.

Symbols

The symbols c_1, c_2 are reserved for use as arbitrary constants in the general solution y_h of the homogeneous equation. Symbols d_1, d_2, d_3, ... are reserved for use in the trial solution y of the non-homogeneous equation. Abbreviations: c = constant, d = determined.

Superposition

The relation $y = y_h + y_p$ suggests solving $ay'' + by' + cy = f(x)$ in two stages:

(a) Apply the linear constant coefficient equation **recipe** to find y_h.

(b) Apply the **basic trial solution method** to find y_p.

- We expect to find two arbitrary constants c_1, c_2 in the solution y_h, but in contrast, no arbitrary constants appear in y_p.
- Calling d_1, d_2, d_3, ... *undetermined* coefficients is misleading, because in fact they are eventually *determined*.

Fixup rule II

The rule predicts the corrected trial solution y without having to substitute y into the differential equation.

- Write down y_h, the general solution of homogeneous equation $ay'' + by' + cy = 0$, having arbitrary constants c_1, c_2. Create the corrected trial solution y iteratively, as follows.
- Cycle through each term $d_p A$, where A is a atom. If A is also an atom appearing in y_h, then multiply $d_p A$ and each **related atom** term $d_q B$ by x. Other terms appearing in y are unchanged.
- Repeat until each term $d_p A$ has atom A distinct from all atoms appearing in homogeneous solution y_h. The modified expression y is called the **corrected trial solution**.

Fixup rule III

The rule predicts the corrected trial solution y without substituting it into the differential equation. This iterative algebraic method uses the atom list of the homogeneous equation to create y.

- Write down the roots of the characteristic equation. Let L denote the list of distinct atoms for these roots.
- Cycle through each term d_pA, where A is a atom. If A appears in list L, then multiply d_pA and each **related atom** term d_qB by x. Other terms appearing in y are unchanged.
- Repeat until the atom A in an arbitrary term d_pA of y does not appear in list L.[a] The modified expression y is called the **corrected trial solution**.

[a]The number s of repeats for initial term d_pA equals the multiplicity of the root r which created atom A in list L.

Definition of function atomRoot

- $\mathrm{atomRoot}(x^j e^{rx}) = r$ for r real.
- $\mathrm{atomRoot}(x^j e^{ax}\cos bx) = \mathrm{atomRoot}(x^j e^{ax}\sin bx) = a + ib$.

Fixup rule IV

The rule predicts the corrected trial solution y without substituting it into the differential equation. This algebraic method uses the roots of the characteristic equation to correct y.

- Write down the roots of the characteristic equation as a list R, according to multiplicity.
- Subdivide trial solution y into groups G of related atoms, by collecting terms and inserting parentheses.
- If a group G contains an atom A with $r = \mathrm{atomRoot}(A)$ in list R, then multiply all terms of G by x^s, where s is the multiplicity of root r.
- Repeat the previous step for all groups G in y. The modified expression y is called the **corrected trial solution**.