Applied Differential Equations 2280
Sample Final Exam
Wednesday, 6 May 2009, 7:30-10:15am

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for the general solution $y(x)$ in the equation $y' = 2\cot x + \dfrac{1250x^3}{1+25x^2} + x\ln(1+x^2)$.

[The required integration talent includes basic formulae, integration by parts, substitution and college algebra.]

Answer:

$$y = 2\ln(\sin(x)) + \frac{49}{2}x^2 - \ln\left(1+25x^2\right) + 1/2\left(1+x^2\right)\ln\left(1+x^2\right) - 1/2 + C$$

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2. (Separable Equation Test)

The problem $y' = f(x,y)$ is said to be separable provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [75%] Check (X) the problems that can be put into separable form, but don't supply any details.

☐	$y' = -y(2xy+1) + (2x+3)y^2$	☐	$yy' = xy^2 + 5x^2y$
☐	$y' = e^{x+y} + e^y$	☐	$3y' + 5y = 10y^2$

(b) [25%] State a test which can verify that an equation is not separable. Use the test to verify that $y' = x + \sqrt{|xy|}$ is not separable.

Answer:

(a) $yy' = xy^2 + 5x^2y$ is not separable, but the other three are separable.

(b) Test: f_y/f not independent of x implies not separable.

Let $f = x + \sqrt{|xy|}$ and assume $x > 0$. Then $y > 0$ and $f = x + \sqrt{xy}$. We have $f_y = 1/y$ and $f_y/f = y/(x+\sqrt{xy})$ depends on x, so the DE is not separable.

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3. (Solve a Separable Equation)

Given $y^2y' = \dfrac{2x^2+3x}{1+x^2}\left(\dfrac{125}{64} - y^3\right)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly.

Answer:

(a) $y = 5/4$

(b)

$$-\frac{1}{3}\ln|125 - 64y^3| = 2x + \frac{3}{2}\ln(1+x^2) - 2\arctan(x) + c$$

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4. (Linear Equations)

(a) [60%] Solve $2v'(t) = -32 + \dfrac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.

(b) [30%] Solve $2\sqrt{x+2}\,\dfrac{dy}{dx} = y$. The answer contains symbol c.

(c) [10%] The problem $2\sqrt{x+2}\,y' = y - 5$ can be solved using the answer y_h from part (b) plus superposition $y = y_h + y_p$. Find y_p. Hint: If you cannot write the answer in a few seconds, then return here after finishing all problems on the exam.

Answer:

(a) $v(t) = -24t - 8$

(b) $y(x) = Ce^{\sqrt{x+2}}$

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$dx/dt = 1000\left(2 - \sqrt[5]{x}\right)^3(2+3x)(9x^2-4)^8.$$

Expected in the diagram are equilibrium points and signs of x' (or flow direction markers $<$ and $>$).

(b) [40%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

(c) [10%] Outline how to solve for non-equilibrium solutions, without doing any integrations or long details.

Answer:

(a) and (b) See a handwritten exam solution for a similar problem on midterm 1.
(c) Put the DE into the form $y'/G(y) = F(x)$ and then apply the method of quadrature.

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6. (ch3)
(a) Solve for the general solutions:

(**a.1**) [25%] $y'' + 4y' + 4y = 0$,

(**a.2**) [25%] $y^{vi} + 4y^{iv} = 0$,

(**a.3**) [25%] Char. eq. $r(r-3)(r^3 - 9r)^2(r^2 + 4)^3 = 0$.

(b) Given $6x''(t) + 7x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 7$, $k = 2$, solve the differential equation [15%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a physical model drawing the meaning of constants m, c, k [5%].

Answer:
(a)
1: $r^2 + 4r + 4 = 0$, $y = c_1y_1 + c_2y_2$, $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.
2: $r^{iv} + 4r^2 = 0$, roots $r = 0, 0, 2i, -2i$. Then $y = c_1e^{0x} + c_2xe^{0x} + c_3 \cos 2x + c_4 \sin 2x$.
3: Write as $r^3(r-3)^3(r+3)^2(r^2+4)^3 = 0$. Then y is a linear combination of the atoms 1, x, x^2, e^{3x}, xe^{3x}, x^2e^{3x}, e^{-3x}, xe^{-3x}, $\cos 2x$, $x \cos 2x$, $x^2 \cos 2x$, $\sin 2x$, $x \sin 2x$, $x^2 \sin 2x$.
Part (b)
Use $6r^2 + 7r + 2 = 0$ and the quadratic formula to obtain roots $r = -1/2, -2/3$. Then $x(t) = c_1e^{-t/2} + c_2e^{-2t/3}$. This is over-damped. The illustration shows a spring, dampener and mass with labels k, c, m, x and the equilibrium position of the mass.

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7. (ch3)
Determine for $y^{vi} + y^{iv} = x + 2x^2 + x^3 + e^{-x} + x \sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Answer:
The homogeneous solution is a linear combination of the atoms 1, x, x^2, x^3, $\cos x$, $\sin x$ because the characteristic polynomial has roots 0, 0, 0, 0, i, $-i$.
$\boxed{1}$ An initial trial solution y is constructed for atoms 1, x, e^{3x}, e^{-3x}, $\cos x$, $\sin x$ giving

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 + d_2x + d_3x^2 + d_4x^3, \\ y_2 &= d_5 \cos x + d_6 x \cos x, \\ y_3 &= d_7 \sin x + d_8 x \sin x, \\ y_4 &= d_9 e^{-x}. \end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by specialization of constants, all derivatives of the right side of the differential equation.

2 The correction rule is applied individually to each of y_1, y_2, y_3, y_4.

The result is the **shortest trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 x^2 + d_2 x^3 + d_3 x^4 + d_4 x^5, \\ y_2 &= d_5 x \cos x + d_6 x^2 \cos x, \\ y_3 &= d_7 x \sin x + d_8 x^2 \sin x, \\ y_4 &= d_9 e^{-x}. \end{aligned}$$

Some facts:

- The number of terms in each of y_1 to y_4 is unchanged.
- If an atom of the homogeneous equation appears in a group, then it is removed. The crossed-out term is replaced by adding another term on **the end** of that group.
- Suppose a group has base atom A. The number s of crossed-out terms for this group is exactly the number of atoms of the homogeneous equation having base atom A.
  This number s is the corresponding root multiplicity for base atom A in the characteristic equation. The value s appears in the Edwards-Penney table containing the mystery factor x^s.

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8. (ch3)
(a) [50%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 4x' + 6x = 10\cos(2t)$.
(b) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' + 3y' + 2y = xe^{2x}$.

Answer:
(a) Undetermined coefficients for $x'' + 4x' + 6x = 10\cos(2t)$.
We solve $x'' + 4x' + 6x = 0$. The characteristic equation roots are $-2 \pm \sqrt{2}i$ and the atoms are $x_1 = e^{-2t}\cos\sqrt{2}t$, $x_2 = e^{-2t}\sin\sqrt{2}t$.
The trial solution is computed from $f = 10\cos 2t$. We find all atoms in the derivative list $f, f', f'' \ldots$, then take a linear combination to form the trial solution $x = d_1 \cos 2t + d_2 \sin 2t$. No corrections are needed, because none of these terms are solutions of the homogeneous equation $x'' + 4x' + 6x = 0$.
Substitute the trial solution to obtain the answers $d_1 = 5/17$, $d_2 = 20/17$. Cramer's rule was used to solve for the unknowns d_1, d_2.
The unique periodic solution x_{SS} is extracted from the general solution $x = x_h + x_p$ by crossing out all negative exponential terms (terms which limit to zero at infinity). Because $x_h(t)$ contains atoms x_1, x_2 (see above) containing a negative exponential factor, then

the steady state solution is

$$x_{\mathsf{ss}} = \frac{5}{17}\cos 2t + \frac{20}{17}\sin 2t.$$

(b) Variation of parameters for $y'' + 3y' + 2y = xe^{2x}$.
We solve $y'' + 3y' + 2y = 0$. The characteristic equation roots are $-2, -1$ and the atoms are $y_1 = e^{-2x}$, $y_2 = e^{-x}$.
Compute the Wronskian $W = y_1 y_2' - y_1' y_2 = e^{-x}$. Then for $f(x) = xe^{2x}$,

$$y_p(x) = \left(\int y_2 \frac{-f}{W} dx\right) y_1(x) + \left(\int y_1 \frac{f}{W} dx\right) y_2(x).$$

Substitution of y_1, y_2, W, f gives

$$y_p(x) = \left(\int e^{-x} \frac{-xe^{2x}}{e^{-3x}} dx\right) y_1(x) + \left(\int e^{-2x} \frac{xe^{2x}}{e^{-3x}} dx\right) y_2(x) = \frac{-7e^{2x}}{144} + \frac{xe^{2x}}{12}.$$

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9. (ch5)
The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix},$$

then
(a) [75%] Display eigenanalysis details for A.
(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Answer:
(1): The eigenpairs are

$$\left(4, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(6, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(7, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right).$$

An expected detail is the cofactor expansion of $\det(A - \lambda I)$ and factoring to find eigenvalues $4, 6, 7$. Eigenvectors should be found by a sequence of swap, combo, mult operations on the augmented matrix, followed by taking the partial ∂_{t_1} on invented symbol t_1 in the general solution to compute the eigenvector.
(2): The eigenanalysis method for $\mathbf{x}' = A\mathbf{x}$ implies

$$\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{7t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

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10. (ch5)

(a) [20%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -2 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to Putzer's spectral formula. Don't expand matrix products, in order to save time. However, please compute all three coefficient functions r_1, r_2, r_3.

(c) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton Method. In particular, display the equations that determine the three vectors in the general solution. To save time, **don't solve for the three vectors**.

(d) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Eigenanalysis Method. To save time, find one eigenvector explicitly, but **don't solve for the last two eigenvectors**.

(e) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to Laplace's Method. To save time, use symbols for partial fraction constants and **leave the symbols unevaluated**.

Answer:

(a) **Eigenvalue Calculation**

Subtract λ from the diagonal elements of A to obtain matrix $B = A - \lambda I$, then expand $\det(B)$ by cofactors to obtain the characteristic polynomial. The roots are the eigenvalues $\lambda = 2, 3, 5$.

(b) **Putzer Method for the Exponential Matrix**

Let

$$P = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 5 \end{pmatrix}.$$

Define functions r_1, r_2, r_3 to be the components of the vector solution $\mathbf{r}(t)$ of the initial value problem

$$\mathbf{r}' = P\mathbf{r}, \quad \mathbf{r}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In expanded scalar form the equations are

$$\begin{aligned} r_1' &= 2r_1, \quad r_1(0) = 1 \\ r_2' &= 3r_2 + r_1, \quad r_2(0) = 0 \\ r_3' &= 5r_3 + r2, \quad r_3(0) = 0. \end{aligned}$$

Solving by the linear integrating factor method

$$r_1 = e^{2t}, \quad r_2 = e^{3t} - e^{2t}, \quad r_3 = \frac{1}{3}e^{2t} - \frac{1}{2}e^{3t} + \frac{1}{6}e^{4t}.$$

Define the Putzer projections $P_1 = I, \quad P_2 = A - 2I, \quad P_3 = (A-2I)(A-3I)$. Then $e^{At} = r_1(t)P1 + r_2(t)P_2 + r_3(t)P_3$ and $\mathbf{u}(t) = e^{At}\mathbf{u}_0$ implies

$$\mathbf{u} = (r_1 I + r_2(A-2I) + r_3(A-2I)(A-3I))\,\mathbf{u}_0$$

(c) **Cayley-Hamilton Method**

The eigenvalues $2, 3, 5$ from (a) are used to create the list of atoms e^{2t}, e^{3t}, e^{5t}. Then the Cayley-Hamilton method implies there are constant vectors $\vec{\mathbf{c}}_1$, $\vec{\mathbf{c}}_3$, $\vec{\mathbf{c}}_3$ which depend on $\vec{\mathbf{u}}(0)$ and A such that

$$\vec{\mathbf{u}}(t) = e^{2t}\vec{\mathbf{c}}_1 + e^{3t}\vec{\mathbf{c}}_2 + e^{5t}\vec{\mathbf{c}}_3.$$

The determining equations are formed from differentiation of this formula two times, then replace $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, $\vec{\mathbf{u}}'' = A\vec{\mathbf{u}}' = AA\vec{\mathbf{u}}$. Finally, remove t from the three equations by setting $t = 0$, and define $\vec{\mathbf{u}}_0 = \vec{\mathbf{u}}(0)$. Then the three equations are

$$\begin{array}{rcrcrcr} \vec{\mathbf{u}}_0 & = & \vec{\mathbf{c}}_1 & + & \vec{\mathbf{c}}_2 & + & \vec{\mathbf{c}}_3 \\ A\vec{\mathbf{u}}_0 & = & 2\vec{\mathbf{c}}_1 & + & 3\vec{\mathbf{c}}_2 & + & 5\vec{\mathbf{c}}_3 \\ A^2\vec{\mathbf{u}}_0 & = & 4\vec{\mathbf{c}}_1 & + & 9\vec{\mathbf{c}}_2 & + & 25\vec{\mathbf{c}}_3 \end{array}$$

This ends the solution to the problem. We continue, solving for the vectors $\vec{\mathbf{c}}_j$, just to illustrate how it is done. The matrix of coefficients

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{pmatrix}$$

and its transpose matrix $B = C^T$ give a formal relation

$$\mathbf{aug}(\vec{\mathbf{u}}_0, A\vec{\mathbf{u}}_0, A^2\vec{\mathbf{u}}_0) = \mathbf{aug}(\vec{\mathbf{c}}_1, \vec{\mathbf{c}}_2, \vec{\mathbf{c}}_3)B.$$

Multiplying this relation by B^{-1} gives

$$\mathbf{aug}(\vec{\mathbf{c}}_1, \vec{\mathbf{c}}_2, \vec{\mathbf{c}}_3) = \mathbf{aug}(\vec{\mathbf{u}}_0, A\vec{\mathbf{u}}_0, A^2\vec{\mathbf{u}}_0)B^{-1}.$$

Then disassembling the formal matrix multiply implies

$$\begin{array}{rcrcrcr} \vec{\mathbf{c}}_1 & = & 5\vec{\mathbf{u}}_0 & - & \frac{8}{3}A\vec{\mathbf{u}}_0 & + & \frac{1}{3}A^2\vec{\mathbf{u}}_0 \\ \vec{\mathbf{c}}_2 & = & -5\vec{\mathbf{u}}_0 & + & \frac{7}{2}A\vec{\mathbf{u}}_0 & - & \frac{1}{2}A^2\vec{\mathbf{u}}_0 \\ \vec{\mathbf{c}}_3 & = & 5\vec{\mathbf{u}}_0 & - & \frac{5}{6}A\vec{\mathbf{u}}_0 & + & \frac{1}{6}A^2\vec{\mathbf{u}}_0 \end{array}$$

The matrix of coefficients is

$$\begin{pmatrix} 5 & -\frac{8}{3} & \frac{1}{3} \\ -5 & \frac{7}{2} & -\frac{1}{2} \\ 1 & -\frac{5}{6} & \frac{1}{6} \end{pmatrix} = \left(B^{-1}\right)^T = C^{-1}!$$

This fact, that solving for $\vec{\mathbf{c}}_1$, $\vec{\mathbf{c}}_1$, $\vec{\mathbf{c}}_1$ in the displayed equations reduces to inverting the matrix of coefficients, can be used as a shortcut in the Cayley-Hamilton method. (d)

Eigenanalysis Method
For matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

the eigenpairs are computed to be

$$\left(2, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right).$$

Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ has general solution

$$\vec{\mathbf{u}}(t) = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(e) **Laplace's Method**
The start is the Laplace resolvent formula for matrix differential equation $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$.

$$(sI - A)\,\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}_0.$$

This formula expands to

$$\begin{pmatrix} s-4 & -1 & 1 \\ -1 & s-4 & 2 \\ 0 & 0 & s-2 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \\ \mathcal{L}(z) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where symbols a, b, c are arbitrary constants for the initial data $\vec{\mathbf{u}}_0$. Let W denote the coefficient matrix. Then the inverse of W can be computed using the adjugate formula $W^{-1} = \mathbf{adj}(W)/\det(W)$. The answer for the inverse is

$$W^{-1} = \frac{1}{(s-5)(s-2)(s-3)} \begin{pmatrix} s^2 - 6\,s + 8 & s-2 & -s+2 \\ s-2 & s^2 - 6\,s + 8 & -2\,s + 7 \\ 0 & 0 & s^2 - 8\,s + 15 \end{pmatrix}$$

True, this formula can be derived and then followed by inverse Laplace methods to obtain an answer in variable t. However, we already know the outcome, because this matrix is the Laplace of the exponential matrix e^{At}. The exponential matrix formula was already derived in (b) above. Expanding the matrix multiplies and collecting terms gives the final answer

$$W^{-1} = \mathcal{L}\left(e^{At}\right) = \frac{1}{2}\mathcal{L} \begin{pmatrix} \mathrm{e}^{5\,t} + \mathrm{e}^{3\,t} & \mathrm{e}^{5\,t} - \mathrm{e}^{3\,t} & -\mathrm{e}^{5\,t} + \mathrm{e}^{3\,t} \\ \mathrm{e}^{5\,t} - \mathrm{e}^{3\,t} & \mathrm{e}^{5\,t} + \mathrm{e}^{3\,t} & -\mathrm{e}^{5\,t} + 2\,\mathrm{e}^{2\,t} - \mathrm{e}^{3\,t} \\ 0 & 0 & 2\,\mathrm{e}^{2\,t} \end{pmatrix}$$

Canceling the $\mathcal{L}$ with Lerch's Theorem implies the same answer as found in part (b), which is

$$\vec{\mathbf{u}}(t) = e^{At}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t}+e^{3t} & e^{5t}-e^{3t} & -e^{5t}+e^{3t} \\ e^{5t}-e^{3t} & e^{5t}+e^{3t} & -e^{5t}+2\,e^{2t}-e^{3t} \\ 0 & 0 & 2\,e^{2t} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

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11. (ch5) Do enough to make 100%

(a) [50%] The eigenvalues are 4, 6 for the matrix $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$. Show details from either the eigenanalysis method or the Laplace method.

(b) [50%] Using the same matrix A from part (a), display the solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton Method. To save time, write out the system to be solved for the two vectors, and then stop, without solving for the vectors.

(c) [50%] Using the same matrix A from part (a), compute the exponential matrix e^{At} by any known method, for example, the formula $e^{At} = \Phi(t)\Phi^{-1}(0)$, or Putzer's formula.

Answer:

(a) Eigenanalysis method

The eigenpairs of A are

$$\left(4, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \quad \left(6, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

which implies the eigenanalysis general solution

$$\mathbf{u}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b)Cayley-Hamilton method

Then $\mathbf{u}(t) = e^{4t}\vec{\mathbf{c}}_1 + e^{6t}\vec{\mathbf{c}}_2$ for some constant vectors $\vec{\mathbf{c}}_1$, $\vec{\mathbf{c}}_2$ that depend on $\vec{\mathbf{u}}(0)$ and A. Differentiate this equation once and use $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, then set $t = 0$. The resulting system is

$$\begin{aligned} \vec{\mathbf{u}}_0 &= e^0\vec{\mathbf{c}}_1 + e^0\vec{\mathbf{c}}_2 \\ A\vec{\mathbf{u}}_0 &= 4e^0\vec{\mathbf{c}}_1 + 6e^0\vec{\mathbf{c}}_2 \end{aligned}$$

(c) Putzer Method

The result is $e^{At} = e^{4t}I + \frac{e^{4t}-e^{6t}}{4-6}(A - 4I)$. Functions r_1, r_2 are computed from $r_1' = 4r_1$, $r_1(0) = 1$, $r_2' = 6r_2 + r_1$, $r_2(0) = 0$.

$$e^{At} = \frac{1}{2}\begin{pmatrix} e^{4t}+e^{6t} & e^{6t}-e^{4t} \\ e^{6t}-e^{4t} & e^{4t}+e^{6t} \end{pmatrix}.$$

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12. (ch5) Do both

(a) [50%] Display the solution of $\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, using any method that applies.

(b) [50%] Display the variation of parameters formula for the system below. Then integrate to find $\mathbf{u}_p(t)$ for $\mathbf{u}' = A\mathbf{u}$.

$$\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}.$$

Answer:
(a) **Resolvent method**
The resolvent equation $(sI - A)\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}(0)$ is the system

$$\begin{pmatrix} s-2 & 0 \\ -1 & s-2 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The system is solved by Cramer's rule for unknowns $\mathcal{L}(x)$, $\mathcal{L}(y)$ to obtain

$$\mathcal{L}(x) = \frac{0}{(s-2)^2}, \quad \mathcal{L}(y) = \frac{s-2}{(s-2)^2}.$$

The backward Laplace table implies

$$x(t) = 0, \quad y(t) = e^{2t}.$$

Best method. Look at the equations as scalar equations $x' = 2x$, $x(0) = 0$ and $y' = x + 2y$, $y(0) = 1$. Clearly $x(t) = 0$ and then $y' = 0 + 2y$, $y(0) = 1$ implies $y(t) = e^{2t}$.
(b) Putzer Method
Putzer's exponential formula gives

$$e^{At} = e^{2t}I + te^{2t}(A - 2I) = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix}.$$

Then $\vec{\mathbf{u}}_p(t) = e^{At} \int_0^t e^{-Au} \begin{pmatrix} e^{2u} \\ 0 \end{pmatrix} du = e^{At} \int_0^t \begin{pmatrix} 1 \\ -u \end{pmatrix} du = \begin{pmatrix} te^{2t} \\ t^2e^{2t}/2 \end{pmatrix}$.

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13. (ch6)
(a) Define *asymptotically stable equilibrium* for $\mathbf{u}' = \mathbf{f}(\mathbf{u})$, a 2-dimensional system.
(b) Give examples of 2-dimensional systems of type saddle, spiral, center and node.

(c) Give a 2-dimensional predator-prey example $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ and explain the meaning of the variables in the model.

Answer:

(a) Definition

An equilibrium point $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ is **asymptotically stable** at $t = \infty$ provided it is stable and in addition $\lim \vec{\mathbf{u}}(t) = \vec{\mathbf{u}}_0$ for all solutions $\vec{\mathbf{u}}(t)$ with $\|\vec{\mathbf{u}}(0) - \vec{\mathbf{u}}_0\|$ sufficiently small.

An equilibrium point $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ is **stable** at $t = \infty$ provided for each $\epsilon > 0$ there corresponds a number $\delta > 0$, depending on ϵ, such that $\|\vec{\mathbf{u}}(0) - \vec{\mathbf{u}}_0\| < \delta$ implies $\vec{\mathbf{u}}(t)$ exists for $0 \le t < \infty$ and for all such t-values $\|\vec{\mathbf{u}}(t) - \vec{\mathbf{u}}_0\| < \epsilon$.

(b) Examples

The roots are considered for each type, to give the correct geometric picture. For instance, the example $x = e^t$, $y = e^{-t}$ traces out one branch of a saddle. Here's the answers:

$$\begin{aligned}
A_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && \text{saddle} \quad \text{roots} = 1,\ -1 \\
A_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && \text{node} \quad \text{roots} = 1,\ 1 \\
A_3 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} && \text{center} \quad \text{roots} = i,\ -i \\
A_4 &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} && \text{spiral} \quad \text{roots} = -1+i,\ -1-i
\end{aligned}$$

(c) Predator-Prey

An example is

$$\begin{aligned}
x' &= (1 - x - y)x, \\
y' &= (2 - y + x)y.
\end{aligned}$$

Then x is the prey and y is the predator. Without the interaction terms, each population would change according to a logistic equation, with carrying capacities of 1 and 2, respectively, for x and y. There are four equilibria, three of which are extinction states and one that represents the ideal population sizes, about which the real populations oscillate.

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14. (ch6)

Find the equilibrium points of $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$ and classify the linearizations as node, spiral, center, saddle. What classifications can be deduced for the nonlinear system?

Answer:

The equilibria are constant solutions, which are found from the equations

$$\begin{aligned}
0 &= (14 - x/2 - y)x \\
0 &= (16 - y/2 - x)y
\end{aligned}$$

Considering when a zero factor can occur leads to the four equilibria $(0,0)$, $(0,32)$, $(28,0)$, $(12,8)$. The last equilibrium comes from solving the system of equations

$$\begin{aligned} x/2+y &= 14 \\ x+y/2 &= 16 \end{aligned}$$

Linearization

The Jacobian matrix J is the augmented matrix of partial derivatives $\partial_x\vec{\mathbf{F}}$, $\partial_y\vec{\mathbf{F}}$ (column vectors) computed from

$$\vec{\mathbf{f}}(x,y) = \begin{pmatrix} 14x - x^2/2 - yx \\ 16y - y^2/2 - xy \end{pmatrix}.$$

Then

$$J(x,y) = \begin{pmatrix} 14-x-y & -x \\ -y & 16-y-x \end{pmatrix}.$$

The four matrices below are $J(x,y)$ when (x,y) is replaced by an equilibrium point. Included in the table are the roots of the characteristic equation for each matrix and its classification based on the roots. No book was consulted for the classifications. The idea in each is to examine the limits at $t=\pm\infty$, then eliminate classifications. No matrix has complex eigenvalues, and that eliminates the center and spiral. The first three are stable at either $t = infty$ or $t = infty$, which eliminates the saddle and leaves the node as the only possible classification.

$A_1 = J(0,0)$	$= \begin{pmatrix} 14 & 0 \\ 0 & 16 \end{pmatrix}$	$r = 14, 16$	node
$A_2 = J(0,32)$	$= \begin{pmatrix} -18 & 0 \\ -32 & -16 \end{pmatrix}$	$r = -18, -16$	node
$A_3 = J(28,0)$	$= \begin{pmatrix} -14 & -28 \\ 0 & -12 \end{pmatrix}$	$r = -14, -12$	node
$A_4 = J(12,8)$	$= \begin{pmatrix} -6 & -12 \\ -8 & -4 \end{pmatrix}$	$r = -5+\sqrt{97}, -5-\sqrt{97}$	saddle

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Some maple code for checking the answers:

```
F:=unapply([14*x-x^2/2-y*x , 16*y-y^2/2 -x*y],(x,y));
Fx:=unapply(map(u->diff(u,x),F(x,y)),(x,y));
Fy:=unapply(map(u->diff(u,y),F(x,y)),(x,y));
Fx(0,0);Fy(0,0);Fx(28,0);Fy(28,0);Fx(0,32);Fy(0,32);Fx(0,32);Fy(0,32);
```

15. (ch6) Do enough to make 100%
(a) [25%] Which of the four types *center, spiral, node, saddle* can be unstable at $t=\infty$? Explain your answer.
(b) [25%] Give an example of a linear 2-dimensional system $\mathbf{u}' = A\mathbf{u}$ with a saddle

at equilibrium point $x = y = 0$, and A is not triangular.
(c) [25%] Give an example of a nonlinear 2-dimensional predator-prey system with exactly four equilibria.
(d) [25%] Display a formula for the general solution of the equation $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{u}$. Then explain why the system has a spiral at $(0,0)$.
(e) [25%] Is the origin an isolated equilibrium point of the $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{u}$? Explain your answer.

Answer:
(a) All except the center, which is stable but not asymptotically stable. All the others correspond to a general solution which can have an exponential factor e^{kt} in each term. If $k > 0$, then the solution cannot approach the origin at $t = \infty$.
(b) Required are characteristic roots like $1, -1$. Let $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Define $A = PBP^{-1}$ where $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\mathbf{u}' = A\mathbf{u}$ has a saddle at the origin, because the characteristic roots of A are still $1, -1$. And $A = \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$ is not triangular.
(c) **Example**: The nonlinear predator-prey system $x' = (x+y-4)x$, $y' = (-x+2y-2)y$ has exactly four equilibrium points $(0,0)$, $(4,0)$, $(0,1)$, $(2,2)$.
(d) The characteristic equation $\det(A - \lambda I) = 0$ is $(1-\lambda)^2 + 1 = 0$ with complex roots $1 \pm i$ and corresponding atoms $e^t \cos t$, $e^t \sin t$. Then the Cayley-Hamilton Method implies

$$\vec{\mathbf{u}}(t) = e^t \cos t \vec{\mathbf{c}}_1 + e^t \sin t \vec{\mathbf{c}}_2.$$

Explanation, why the classification is a spiral. Such solutions containing sine and cosine factors wrap around the origin. This makes it a spiral or a center. Because of the exponential factor e^t, it is asymptotically stable at $t = -\infty$, which disallows a center, so it is a spiral.
(e) No, because $\det(A) = 0$. In this case, $A\mathbf{u} = \mathbf{0}$ has infinitely many solutions, describing a line of equilibria through the origin. This implies the equilibrium point $(0,0)$ is not isolated [you cannot draw a circle about $(0,0)$ which contains no other equilibrium point].

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16. (ch7)
(a) Define the direct Laplace Transform.
(b) Define Heaviside's unit step function.
(c) Derive a Laplace integral formula for Heaviside's unit step function.
(d) Explain Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$.

Answer:

(a) Definition of Direct Laplace Transform

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt.$$

(b) Definition of the Heaviside unit step

$$u(t-a) = \begin{cases} 1 & t \ge a, \\ 0 & t < a. \end{cases}$$

(c) Derivation
We prove the second shifting theorem $\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\mathcal{L}(f(t))$, which includes an integral formula for the Heaviside function.

$$\begin{array}{rcl} \mathcal{L}(u(t-a)f(t-a)) & = & \int_0^\infty u(t-a)f(t-a)e^{-st}dt \\ & = & \int_0^a (\text{integrand})dt + \int_a^\infty (\text{integrand})dt \\ & = & 0 + \int_a^\infty f(t-a)e^{-st}dt \\ & = & \int_0^\infty f(u)e^{-s(a+u)}du \\ & = & e^{-sa}\int_0^\infty f(u)e^{-su}du \\ & = & e^{-as}\mathcal{L}(f(t)) \end{array}$$

Used in the derivation is a change of variable $u = t-a$, $du = dt$. Line 3 uses $u(t-a) = 0$ on the interval $0 \le t \le a$ and $u(t-a) = 1$ on $a \le t < \infty$, which simplifies each integrand. Line 5 observes that factor e^{-sa} in the integrand is a constant relative to u-integration, therefore it can move through the integral sign.
(d) Laplace's method explained.
The first step transforms the equation using the parts formula and initial data to get

$$(s+2)\mathcal{L}(x) = 1 + \mathcal{L}(e^t).$$

The forward Laplace table applies to write, after a division, the isolated formula for $\mathcal{L}(x)$:

$$\mathcal{L}(x) = \frac{1 + 1/(s-1)}{s+2} = \frac{s}{(s-1)(s+2)}.$$

Partial fraction methods imply

$$\mathcal{L}(x) = \frac{a}{s-1} + \frac{b}{s+2} = \mathcal{L}(ae^t + be^{-2t})$$

and then $x(t) = ae^t + be^{-2t}$ by Lerch's theorem. The constants are $a = 1/3$, $b = 2/3$.

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17. (ch7)
(a) Solve $\mathcal{L}(f(t)) = \dfrac{100}{(s^2+1)(s^2+4)}$ for $f(t)$.

(b) Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s-3)}$.

(c) Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}\sin(3t)$.
(d) Find $\mathcal{L}(f)$ where $f(t)$ is the periodic function of period 2 equal to $t/2$ on $0 \le t \le 2$ (sawtooth wave).

Answer:
(a) $\mathcal{L}(f) = \frac{100}{(u+1)(u+4)} = \frac{100/3}{u+1} + \frac{-100/3}{u+4}$ where $u = s^2$. Then $\mathcal{L}(f) = \frac{100}{3}(\frac{1}{s^2+1} - \frac{1}{s^2+4}) = \frac{100}{3}\mathcal{L}(\sin t - \frac{1}{2}\sin 2t)$ implies $f(t) = \frac{100}{3}(\sin t - \frac{1}{2}\sin 2t)$.
(b) $\mathcal{L}(f) = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-3} = \mathcal{L}(a + bt + ce^{3t})$ implies $f(t) = a + bt + ce^{3t}$. The constants, by Heaviside coverup, are $a = -1/9$, $b = -1/3$, $c = 1/9$.
(c) $\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(e^{2t}\sin 3t)$ by the s-differentiation theorem. The first shifting theorem implies $\mathcal{L}(e^{2t}\sin 3t) = \mathcal{L}(\sin 3t)|_{s\to(s-2)}$. Finally, the forward table implies $\mathcal{L}(f) = \frac{d}{ds}\left(\frac{1}{(s-2)^2+9}\right) = \frac{-2(s-2)}{((s-2)^2+9)^2}$.

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18. (ch7)
(a) Solve $y'' + 4y' + 4y = t^2$, $y(0) = y'(0) = 0$ by Laplace's Method.
(b) Solve $x''' + x'' - 6x' = 0$, $x(0) = x'(0) = 0$, $x''(0) = 1$ by Laplace's Method.
(c) Solve the system $x' = x + y$, $y' = x - y + e^t$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

Answer:
(a) Transform to get $\mathcal{L}(x) = \frac{\mathcal{L}(t^2)}{s^2+4s+4}$. Then $\mathcal{L}(x) = \frac{1}{s^2(s+2)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+2} + \frac{d}{(s+2)^2} = \mathcal{L}(a + bt + ce^{-2t} + dte^{-2t})$. The answer is $x(t) = a + bt + ce^{-2t} + dte^{-2t}$. The partial fraction constants are $a = -1/4$, $b = 1/4$, $c = 1/4$, $d = 1/4$.
(b) Transform to get $\mathcal{L}(x) = \frac{1}{s^3+s^2-6s} = \frac{1}{s(s-2)(s+3)} = \frac{a}{s} + \frac{b}{s-2} + \frac{c}{s+3} = \mathcal{L}(a + be^{2t} + ce^{-3t})$. Then the answer is $x(t) = a + be^{2t} + ce^{-3t}$. The partial fraction constants are $a = -1/6, b = 1/10, c = 1/15$.

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19. (ch7)
(a) [25%] Solve by Laplace's method $x'' + x = \cos t$, $x(0) = x'(0) = 0$.

(b) [10%] Does there exist $f(t)$ of exponential order such that $\mathcal{L}(f(t)) = \dfrac{s}{s+1}$? Details required.

(c) [15%] Linearity $\mathcal{L}(c_1 f + c_2 g) = c_1\mathcal{L}(f) + c_2\mathcal{L}(g)$ is one Laplace rule. State four other Laplace rules. Forward and backward table entries are not rules, which means $\mathcal{L}(1) = 1/s$ doesn't count.

(d) [25%] Solve by Laplace's resolvent method

$$\begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 2x(t), \end{aligned}$$

with initial conditions $x(0) = -1$, $y(0) = 2$.

(e) [25%] Derive $y(t) = \int_0^t \sin(t-u) f(u)du$ by Laplace transform methods from the forced oscillator problem

$$y''(t) + y(t) = f(t), \quad y(0) = y'(0) = 0.$$

Answer:

(a) Transform to obtain $\mathcal{L}(x) = \frac{s}{(s^2+1)^2}$.

Calculus method. Observe that $\frac{d}{ds}\frac{1}{s^2+1} = \frac{-2s}{(s^2+1)^2}$. Then $\mathcal{L}(x) = -\frac{1}{2}\frac{d}{ds}\frac{1}{s^2+1} = -\frac{1}{2}\frac{d}{ds}\mathcal{L}(\sin t) = -\frac{1}{2}\mathcal{L}((-t)\sin t)$ by the s-differentiation theorem. Finally, $x(t) = \frac{1}{2}t\sin t$.

Convolution method. Write $\mathcal{L}(x) = \mathcal{L}(\sin t)\mathcal{L}(\cos t)$. Apply the convolution theorem to obtain $x(t) = \int_0^t \sin u \cos(t-u)du = \frac{1}{2}t\sin t$. A maple answer check is

```
int(sin(u)*cos(t-u),u=0..t);
```

.

Hand integration uses the trigonometric identity $2\sin(a)\cos(b) = \cos(a-b) - \cos(a+b)$.

(b) No. The limit of the Laplace transform of a function of exponential order is zero as $s \to \infty$.

(c) The possible rules: Linearity, Lerch's cancelation law, parts formula, s-differentiation, first shift theorem, second shift theorem, periodic function formula, convolution theorem, delta function formula, integral theorem.

(d) The resolvent formula $(sI - A)\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}_0$ becomes the $2\times$ system of equations

$$\begin{pmatrix} s-1 & -1 \\ -2 & s-0 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Multiply by the inverse matrix of $(sI - A)$ on the left to obtain

$$\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} s-1 & -1 \\ -2 & s-0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} s-0 & 1 \\ 2 & s-1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

where $\Delta = \det(sI - A) = (s+1)(s-2)$. Then $\mathcal{L}(x) = \frac{2-s}{\Delta} = \frac{-1}{s+1}$, $\mathcal{L}(y) = \frac{2s}{\Delta} = \frac{2s-4}{(s+1)(s-2)} = \frac{2}{s+1}$. Then $x(t) = -e^{-t}$, $y(t) = 2e^{-t}$.

(e) Derive $y(t) = \int_0^t \sin(t-u)f(u)du$

Transform $y'' + y = f$ to get the transfer function relation

$$\mathcal{L}(y(t)) = \frac{1}{s^2+1}\mathcal{L}(f(t)) = \mathcal{L}(\sin t)\mathcal{L}(f(t)).$$

The convolution theorem implies the right side of the equation is $\mathcal{L}(\int_0^t \sin(t-u)f(u)ud)$. Lerch's cancelation law implies $y(t) = \int_0^t \sin(t-u)f(u)du$.

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20. (ch7)

(a) [25%] Solve $\mathcal{L}(f(t)) = \dfrac{10}{(s^2+8)(s^2+4)}$ for $f(t)$.

(b) [25%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2(s+2)}$.

(c) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s-1}{s^2+2s+5}$.

(d) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(t^2 \sin 3t)$$

(e) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \left(\mathcal{L}\left(t^3 e^{9t}\cos 8t\right)\right)\Big|_{s\to s+3}.$$

Answer:

(a) $\mathcal{L}(f(t)) = \frac{10}{u+8}u + 4$ where $u = s^2$. Use Heaviside's coverup method to find the partial fraction expansion

$$\frac{10}{u+8}u + 4 = \frac{-5/2}{u+8} + \frac{5/2}{u+4} = \frac{-5/2}{s^2+8} + \frac{5/2}{s^2+4}.$$

Then $\mathcal{L}(f(t)) = \mathcal{L}\left(-\frac{5}{2}\frac{\sin\sqrt{8}t}{\sqrt{8}} + \frac{5}{2}\frac{\sin 2t}{2}\right)$ implies by Lerch's theorem

$$f(t) = -\frac{5}{2}\frac{\sin\sqrt{8}t}{\sqrt{8}} + \frac{5}{2}\frac{\sin 2t}{2}.$$

(b) Expand the fraction into partial fractions as follows:

$$\mathcal{L}(f) = \frac{s+1}{s^2(s+2)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+2} = \mathcal{L}(a + bt + ce^{-2t}).$$

Then Lerch's theorem implies $f(t) = a + bt + ce^{-2t}$. The partial fraction constants are $a = 1/4, b = 1/2, c = -1/4$.

(d) Because $\frac{d}{ds}\mathcal{L}(g(t)) = \mathcal{L}((-t)g(t))$, then $\mathcal{L}(f) = \mathcal{L}((-t)t^2 \sin 3t)$. Lerch's theorem implies $f(t) = -t^3 \sin 3t$.

(e) The shifting theorem $\mathcal{L}(g(t))|_{s\to(s-a)} = \mathcal{L}(e^{at}g(t))$ is applied to remove the shift on the outside and put e^{-3t} into the Laplace integrand. Then $\mathcal{L}(f(t)) = \mathcal{L}(e^{-3t}t^3e^{9t}\cos 8t)$. Lerch's theorem implies $f(t) = t^3e^{6t}\cos 8t$.
