Math 2250 Maple Project 2: Newton Cooling
January 2009

Due date: See the internet due dates.

References: Edwards-Penney, pages 55–57. Code in `maple` appears in `2250mapleL2-S2009a.txt` at URL `http://www.math.utah.edu/~gustafso/`. This document: `2250mapleL2-S2009a.pdf`.

The Problem and Model. The project is to determine an approximation for the indoor temperature $u(t)$ in an unheated building. The model uses Newton's cooling law, insulation data k and a formula for the ambient outside temperature $A(t)$ (see the background section *infra*):

(1)
$$u'(t) + ku(t) = kA(t),$$
$$u(0) = u_0.$$

Assumptions and Notation.

- Let the daily temperature in Salt Lake City vary from a low of $m = 21$ degrees to a high of $M = 49$ degrees with the low and high at 3am and 3pm, respectively.

- The building is assumed insulated, the average insulation factor being represented by a Newton cooling constant k, usually $0.2 \leq k \leq 0.5$.

- The ambient outside temperature is called $A(t)$ and the inside temperature is called $u(t)$, where t is in hours measured from midnight $t = 0$.

- The number u_0 is the temperature at midnight.

- The model assumes that no heat sources or heat sinks are present inside the building.

- Symbol $u_h(t)$ refers to the general solution of the homogeneous equation $u' + ku = 0$. Symbol $u_p(t)$ stands for a particular solution of $u' + ku = kA(t)$.

Background

Newton Cooling Model. Newton's law of cooling is:

> *The rate of change of the indoor temperature is proportional to the difference between the ambient and indoor temperatures.*

The cooling law implies that du/dt is proportional to $A - u$. Let $A(t)$ be the ambient outside temperature and let k denote the proportionality constant in Newton's cooling law. A model for the indoor temperature $u(t)$ is given by $du/dt = k(A - u)$, $u(0) = u_0$, which can be rearranged as

(2)
$$u'(t) + ku(t) = kA(t), \quad u(0) = u_0.$$

The number k is called the **insulation constant**.

Ambient Temperature Model. Let $M = 49$, $m = 21$, $\omega_0 = \pi/12$. The ambient temperature formula below satisfies $M = \max A(t) = A(15)$, $m = \min A(t) = A(3)$ and $A(t)$ is t-periodic of period 24 hours.

$$
\begin{aligned}
A(t) &= \frac{1}{2}(M + m) - \frac{1}{2}(M - m)\cos\omega_0(t - 3) \\
&= 35 - 14\cos\omega_0(t - 3)
\end{aligned}
$$

For use in `maple`, a function of two variables is used to maintain the variable name ω in displays.

$$AA(t, \omega) = 35 - 14\cos\omega(t - 3)$$

Indoor Temperature $u(t)$. The integrating factor method for linear equations $u' + p(t)u = q(t)$ [Edwards-Penney Section 1.5] applies to find the general solution by these steps:

$u' + ku = kA(t)$	Copy the differential equation. The integrating factor is e^{kt}.
$\dfrac{(e^{kt}u)'}{e^{kt}} = kA(t)$	Integrating factor method.
$(e^{kt}u)' = kA(t)e^{kt}$	Clear fractions.
$e^{kt}u = u_0 + \int_0^t kA(x)e^{kx}\,dx$	Integrate both sides with respect to t. Apply the Fundamental Theorem of Calculus. Use $u(0) = u_0$.
$u = u_0 e^{-kt} + e^{-kt}\int_0^t kA(x)e^{kx}\,dx$	Divide to isolate the *explicit solution u*.

Let $u_h(t) = u_0 e^{-kt}$, a solution of the homogeneous differential equation $u' + ku = 0$. Let $u_p(t) = ke^{-kt}\int_0^t e^{kx}A(x)dx$, a particular solution of the nonhomogeneous differential equation $u' + ku = kA(t)$. Then the indoor temperature $u = u_h + u_p$ depends on the time t, the initial temperature u_0, the insulation constant k and the frequency ω (fixed at $\omega_0 = \pi/12$). Write $u = u(t, u_0, k, \omega)$ to emphasize the dependence. In `maple`, advantages exist for adding the variable name ω, which is later set to value $\omega_0 = \pi/12$. Write u as $\boxed{\texttt{U(t,u0,k,omega)}}$ for use in `maple`.

Steady-state solution. The steady-state solution u_{SS} is an expression obtained from the general solution formula $u = u_h + u_p$ by dropping all terms containing a negative exponential. It depends on t, k and ω. This expression is known to be a solution of the differential equation, independent of any arbitrary constants, which represents the **observed temperature** after a long time. In short, u_{SS} is independent of u_0. Theory provides the following result.

> **Theorem** (Unique Periodic Solution) Assume $p \neq 0$ is a constant and $q(x)$ is continuous and T-periodic $[q(x + T) = q(x)]$. Then the differential equation $\dfrac{dy}{dx} + py = q(x)$ has a unique T-periodic solution $y(x)$ given by the initial value
> $$y(0) = \left(e^{pT} - 1\right)^{-1} \int_0^T e^{pt}q(t)dt.$$

Problem L2.1. (Solution Formulas for u_p and u)

Derive by hand, using `maple` assist for values of integrals, an explicit symbolic formula for $u_p(t)$. Display a final formula for $u = u_h + u_p$ which depends only on t, u_0, k and ω. Check your hand answer for u in `maple`. The only `maple` assist in this problem is `maple` integration and the `maple` answer check.

Problem L2.2. (Steady-State Periodic Solution)

Derive by hand a formula for the steady-state periodic solution u_{SS} of $u' + ku = kA(t)$. There is no `maple` assist in this problem, it is all done by hand from the answer in L2.1. The results:

$$
\begin{aligned}
u_{SS}(t) &= 35 - \frac{14k}{k^2 + \omega_0^2}(k\cos\omega_0(t-3) + \omega_0\sin\omega_0(t-3)), \\
u(t) &= Ce^{-kt} + u_{SS}(t), \\
C &= u_0 - u_{SS}(0).
\end{aligned}
$$

Problem L2.3. (Indoor-Outdoor Variation)

Compare in a `maple` graphic oscillations of the indoor temperature $u(t)$ and the outdoor temperature $A(t)$ over a 48-hour period assuming $k = 0.32$, $u_0 = 74$, $\omega = \pi/12$. Include in the plot four horizontal lines for the maximum and minimum temperatures for each of $u(t)$ and $A(t)$. Report the indoor and outdoor temperature variations from this 48-hour plot (max minus min in each case). Report the time delay between the indoor and outdoor temperatures, using this same plot.

Problem L2.4. (Freezing Pipes)

Assume $\omega = \pi/12$ and the insulation constant k ranges from 0.2 to 0.48. Suppose the inside temperature is 76 degrees at midnight when the furnace is turned off. Report approximate ranges of hours and insulation constants, during the first 72 hours and $0.2 < k < 0.48$, for which the indoor temperature is at or below 30 degrees. Justify your logic used to find the ranges, in a short paragraph. Illustrate with a computer graphic.

Staple this page on top of all hand and maple work.
Sample maple code, problem notes and background are on the next page . . .

Problem Notes

Notes on L2.1

A derivation by hand is expected, which parallels the derivation above, giving a formula for u_h and a formula for u_p. Integration for the u_p formula is referenced to an attached maple worksheet appendix. The answer check references a second attached worksheet.

Because many students want to know how to do the integration, based upon a calculus background, more information is provided here. The integration problem can be solved by hand using the book's integral table:

$$u_p(t) = ke^{-kt} \int_0^t e^{kx} \left(35 - 14 \cos\left(\omega(x-3)\right)\right) dx.$$

A change of variable $u = x - 3$ allows use of integral table entry $\int e^{au} \cos(bu)\, du$. The symbol ω will be set to $\pi/12$, but for simplicity, use symbol ω throughout.

You are asked to use `maple`. Its table lookup method is superior to the hand method above, in that no change of variable is required to obtain the answer. This method is accurate and avoids primary pitfalls in using a book table. A coding example appears below, which moves the factor ke^{-kt} inside the integrand, in order to automatically perform simplifications.

```
# maple integral table lookup
unassign('k','omega','t','x');
integrand:=(35-14*cos(omega*(x-3)))*k*exp(k*x-k*t);
F:=unapply(integrand,(x,t,k,omega));
int(F(x,t,k,omega),x=0..t);
```

The formulas for u_h and u_p are used again in L2.2 in order to derive the steady-state solution. Your answer *must contain* symbols t, u0, k, omega.

The answer check in `maple` is organized as follows. The complications of setting $\omega = \pi/12$ are avoided here by leaving ω as a symbol, since it does not affect the answer check.

```
# Test LHS=RHS for u'+ku=kA.
unassign('t','u0','omega','k'):
uh:=u0*exp(-k*t):
up:=(explicit answer from L2.1):
myANS:=uh+up:
LHS:=diff(myANS,t)+k*myANS:
RHS:=k*(35-14*cos(omega*(t-3))):
simplify(expand(LHS-RHS));
```

A successful test of $LHS = RHS$ produces answer *zero*, or an expression that reduces to zero.

Notes on L2.2

The steady-state solution is derived from the hand-generated symbolic solution $u = u_h + u_p$ in L2.1 by dropping all terms that contain the negative exponential factor e^{-kt}. The answer, where $\omega_0 = \pi/12$:

$$u_{\text{SS}} = 35 - \frac{14k}{k^2 + \omega_0^2}\left(k\cos\omega_0(t-3) + \omega_0 \sin\omega_0(t-3)\right).$$

To *check your answer*, compare it against the formula displayed in L2.2.

To get `maple` to report the above formula, it is essential to evaluate everything with ω as a *symbol*, to wit, use in `maple` the statement unassign('omega'): . **Beware** of writing omega:=Pi/12 , which defines ω to be a constant, unless you undo the effect immediately afterward. Maple's unassign can be used to erase assignments.

Maple is unable to evaluate limit(exp(-k*t),t=infinity) as zero, because the expression contains k, an unevaluated symbol. This is why you are asked to report the steady-state solution manually, by dropping terms from previously obtained answers.

Notes on L2.3

The outside temperature $A(t)$ (`AA(t,omega)` in `maple`) and the solution $u = U(t, u_0, k, \omega)$ obtained in L2.2 can be coded in `maple` as follows:

```
unassign('t','u0','k','omega'):
AA:=unapply(35-14*cos(omega*(t-3)),(t,omega));
uss:=35-(14*k/(k^2+omega^2))*(k*cos(omega*(t-3))+omega*sin(omega*(t-3))):
uss0:=subs(t=0,uss);
U:=unapply((u0-uss0)*exp(-k*t)+uss,(t,u0,k,omega)):
```

The two curves plus a horizontal line $u = 21$ are placed onto the **plot** by this `maple` command:

```
plot({U(t,76,0.32,Pi/12),AA(t,Pi/12),21},t=0..48);
```

Missing curves? Probably, the missing curves are defined to contain an unevaluated variable name, like $\boxed{\texttt{pi}}$ instead of $\boxed{\texttt{Pi}}$. While `Pi` is the constant 3.14159, lowercase symbol `pi` is a variable name: *case* is significant in `maple`.

Curves vanish on the printer? Add $\boxed{\texttt{color=black}}$ to the plot command, e.g., $\boxed{\texttt{plot(sin(x),x=0..Pi,color=black);}}$.

Click a mouse button on the high and low spots in the graphic. Somewhere on the `maple` worksheet the coordinates of the click are displayed (look around on the screen!). This method gives decimal approximations to the max and min values. Use the four values obtained to expand the list of horizontal lines to four lines. The final plot has six curves.

The **indoor temperature variation** is just the maximum minus the minimum, as computed from the plot of $u(t)$. Beware: u has a maximum at $t = 0$. The outdoor temperature variation is computed from the graph of $A(t)$, which is trapped between two horizontal lines for its max and min.

The **time delay** between the indoor and outdoor temperatures is computed as $|T_2 - T_1|$, where $A(T_1) = \max A(t)$ and $u(T_2) = \max u(t)$. Look at the graphic to find sane answers for T_1 and T_2. See the textbook for a more complete discussion of the ideas. Beware: T_1 and T_2 are values for time in the first 72 hours (abscissa values, not ordinate values). The smallest time delay obtainable from the graphic is to be reported, because the transient solution Ce^{-ht} may affect the graph of $u(t) = Ce^{-ht} + u_{SS}(t)$ for a few hours.

Notes on L2.4

A computer algebra assist for this problem uses `maple`'s function `implicitplot`. This function can plot the equation $u(t, 76, k, \pi/12) = 30$ over the domain $0 \le t \le 72$, $0.2 \le k \le 0.48$. From this plot, and the 3D-plot $z = u(x, 76, y, \pi/12)$, the question is easily answered.

```
with(plots): unassign('t','u0','k','omega'):
uh:=u0*exp(-k*t):
up:=(the answer from L2.1):
myANS:=uh+up:
U:=unapply(myANS,(t,u0,k,omega));
implicitplot(U(t,76,k,Pi/12)=30,t=0..72,k=0.2..0.48);
plot3d({U(t,76,k,Pi/12),30},t=0..72,k=0.2..0.48);
```

The relation between the 2D implicit plot and the 3D plot is seen by slicing the 3D plot at height $z = 30$ to obtain a *bread slice* parallel to the xy-plane. The implicit plot depicts the bread slice by projecting it onto the xy-plane. The slice can be visualized in `maple` by grabbing the 3D graphic with the mouse, to perform a rotation, which exposes a view of the underside of the 3D figure. The normal 3D view has the axes reversed, so you should use the mouse tools to orient the 3D graphic to match the 2D bread slice.

Zoom in on the implicit plot by using a smaller time domain, suggested by the larger plot. The zoom should narrow in on a range of k-values and t-values such that the 3D surface is entirely below the plane $z = 30$. This geometry detects freezing in the given range.

Physically, inside temperature 30 degrees is reached a few hours *after* the outside temperature drops below 30 degrees. During 72 hours, there are three such inside temperature drops, verified from the 3D-plot, where z=temperature. For $k < 0.32$, inside temperature 30F is not reached in the first 23 hours. For $0.34 < k < 0.48$, inside temperature 30 degrees is reached between 5.8 and 8.7 hours. Please report answers in this format:

No freezing 0-23 hours, $0.2 < k < 0.32$,
Freezing 5.8-8.7 hours for $0.34 < k < 0.48$,
Freezing 27.2-33.9 hours for $0.2 < k < 0.48$.

Use as many ranges as needed to describe the situation; between 3 and 7 will be required. An answer is correct if it is supported by the graphic.